

# Achievers

# Mathematics

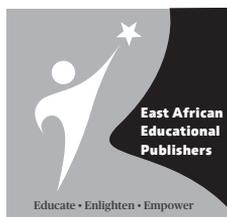
for Rwandan Schools

Senior 4

Teacher's Guide

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# Contents

<b>Introduction</b> .....	<b>1</b>
<b>Topic area: Trigonometry</b> .....	<b>43</b>
<b>Sub-topic area: Trigonometric circle and identities</b> .....	<b>43</b>
Unit 1: Fundamentals of trigonometry .....	43
<b>Topic area: Algebra</b> .....	<b>63</b>
<b>Sub-topic area: Mathematical logic and applications</b> .....	<b>63</b>
Unit 2: Propositional and predicate logic.....	63
<b>Topic area: Algebra</b> .....	<b>79</b>
<b>Sub-topic area: Number and operations</b> .....	<b>79</b>
Unit 3: Binary operations.....	79
<b>Topic area: Algebra</b> .....	<b>86</b>
<b>Sub-topic area: Numbers and operations</b> .....	<b>86</b>
Unit 4: Set $\mathbb{R}$ of real numbers .....	86
<b>Topic area: Algebra</b> .....	<b>94</b>
<b>Sub-topic area: Equation and inequalities</b> .....	<b>94</b>
Unit 5: Linear equations and inequalities .....	94
<b>Topic area: Algebra</b> .....	<b>105</b>
<b>Sub-topic area: Equations and inequalities</b> .....	<b>105</b>
Unit 6: Quadratic equation and inequalities.....	105
<b>Topic area: Analysis</b> .....	<b>114</b>
<b>Sub-topic area: Functions</b> .....	<b>114</b>
Unit 7: Polynomial, rational and irrational functions.....	114
<b>Topic area: Analysis</b> .....	<b>120</b>
<b>Sub-topic area: Limits, differentiation and integration</b> .....	<b>120</b>
Unit 8: Limits of polynomial, rational and irrational functions .....	120
<b>Topic area: Analysis</b> .....	<b>130</b>

<b>Sub-topic area: Limits, differentiation and integration .....</b>	<b>130</b>
Unit 9: Differentiation of polynomials, rational and irrational functions and their application .....	130
<b>Topic area: Linear algebra .....</b>	<b>147</b>
<b>Sub-topic area: Vectors.....</b>	<b>147</b>
Unit 10: Vector space of real numbers .....	147
<b>Topic area: Linear algebra .....</b>	<b>154</b>
<b>Sub-topic area: Linear transformation in 2D.....</b>	<b>154</b>
Unit 11: Concepts and operations on linear transformation in 2D .....	154
<b>Topic area: Linear algebra .....</b>	<b>162</b>
<b>Sub-topic area: Linear transformation in 2D.....</b>	<b>162</b>
Unit 12: Matrices of and determinants of order 2.....	162
<b>Topic area: Geometry .....</b>	<b>168</b>
<b>Sub-topic area: Plane geometry.....</b>	<b>168</b>
Unit 13: Points, straight lines and circles in 2D .....	168
<b>Topic area: Statistics and probability.....</b>	<b>180</b>
<b>Sub-topic area: Descriptive statistics.....</b>	<b>180</b>
Unit 14: Measures of dispersions .....	180
<b>Topic area: Statistics and probability.....</b>	<b>186</b>
<b>Sub-topic area: Combinatorial analysis and probability .....</b>	<b>186</b>
Unit 15: Combinatorics .....	186
<b>Topic area: Statistics and probability.....</b>	<b>194</b>
<b>Sub-topic area: Combinatorial analysis and probability .....</b>	<b>194</b>
Unit 16: Elementary probability.....	194
<b>Answers to Practice Tasks .....</b>	<b>202</b>
<b>References .....</b>	<b>207</b>

# Introduction

The Achievers Mathematics Senior 4 Teacher’s Guide has been designed in response to the new change from a knowledge-based curriculum to a competence-based curriculum. The content in the book has been organised in such a way that it meets the teacher’s requirements in guiding learners to acquire practical skills, competences and attitudes and values a learner should have at the end of each learning unit.

Mathematics is the study of human beings and their physical and social environment. It focuses on equipping learners with knowledge, competences and attitudes and values that help them to live and bring positive change in their environment.

It also focuses on nurturing the young generation into being responsible and productive citizens of their country.

## Importance of Mathematics

Mathematics plays an important role in society through abstraction and logic, counting, calculation, measurement, systematic study of shapes and motion. It is also used in natural sciences, engineering, medicine, finance and social sciences. The applied mathematics like statistics and probability play an important role in game theory, in the national census process, in scientific research, etc. In addition, some cross-cutting issues such as financial awareness are incorporated into some of the mathematics units to improve social and economic welfare of the Rwandan society.

Mathematics is key to the Rwandan education goals of developing a knowledge-based and technology-led economy since it provides learners all required knowledge and skills to be used in different learning areas. Therefore, Mathematics is an important subject as it supports other subjects. This new curriculum will address gaps in the current Rwandan education system which lacks appropriate skills and attitudes.

## ***Mathematics and learners***

Learners need enough basic mathematical competencies to be effective members of the Rwandan society, including the ability to estimate, measure, calculate, interpret statistics, assess probabilities, and read the commonly used mathematical representations and graphs. For example, reading or listening to the news requires some of these competencies, and citizenship requires being able to interpret critically the information one receives.

Therefore, Mathematics equips learners with knowledge, skills and attitudes necessary to enable them to succeed in an era of rapid technological growth and socio-economic development. Mastery of basic Mathematical ideas and operations makes learners being confident in problem-solving. It enables the learners to be systematic, creative and self confident in using mathematical language and techniques to reason, think critically and develop imagination, initiative and flexibility of mind.

Mathematics has a high profile at all levels of study where learning needs to include practical problem-solving activities with opportunities for learners to plan their own investigations and develop their confidence towards Mathematics.

All the above skills will help a learner to grow and develop into a self-reliant citizen.

## Competencies

Competence is defined as the ability to perform a particular task successfully, resulting from having gained an appropriate combination of knowledge, skills and attitudes

The Mathematics syllabus gives the opportunity to learners to develop different competencies, including the generic competencies.

Basic competencies are addressed in the stated broad subject competences in objectives highlighted and in each of units of learning. The generic and basic competences that must be emphasized and reflected in the learning process are briefly described below and teachers will ensure that learners are exposed to tasks that help them acquire the skills.

## Generic competencies and values

**Critical and problem-solving skills:** Learners use different techniques to solve mathematical problems related to real-life situations. They are engaged in mathematical thinking; they construct, symbolize, apply and generalize mathematical ideas.

The acquisition of such skills will help learners to think imaginatively and broadly to evaluate and find solutions to problems encountered in all situations.

**Creativity and innovation:** The acquisition of such skills will help learners to take initiatives and use imagination beyond knowledge provided to generate new ideas and construct new concepts. Learners improve these skills through Mathematics contests and competitions, etc.

**Research:** This will help learners to find answers to questions based on existing information and concepts and to explain phenomena based on findings from information gathered.

**Communication in official languages:** Learners communicate effectively their findings through explanations, construction of arguments and drawing relevant conclusions.

Teachers will ensure the proper use of the language of instruction by learners which will help them to communicate clearly and confidently and convey ideas effectively through speaking and writing and using the correct language structure and relevant vocabulary.

**Cooperation, interpersonal management and life skills:** Learners are engaged in cooperative learning groups to promote higher achievement than do competitive and individual work.

This will help them to cooperate with others as a team in whatever task assigned and to practice positive ethical moral values and respect for the rights, feelings and views of others. Perform practical activities related to environmental conservation and protection. Advocating for personal, family and community health, hygiene and nutrition and responding creatively to the variety of challenges encountered in life.

**Lifelong learning:** The acquisition of such skills will help learners to update knowledge and skills with minimum external support and to cope with evolution of knowledge advances for personal fulfillment in areas that need improvement and development.

### **Broad mathematics competencies**

During and at the end of learning process, the learner can:

1. Use correctly specific symbolism of the fundamental concepts in Mathematics
2. Develop clear, logical, creative and coherent thinking
3. Apply acquired knowledge in Mathematics in solving problems encountered in everyday life
4. Use the acquired concepts for easy adaptation in the study of other subjects
5. Deduce correctly a given situation from a picture and/or a well drawn out basic mathematical concepts and use them correctly in daily life situations
6. Read and interpret a graph

7. Use acquired mathematical skills to develop work spirit, team work, self-confidence and time management without supervision
8. Use ICT tools to explore Mathematics (examples: calculators, computers, mathematical software).

## **Approach used in teaching Mathematics**

### **Methodological approach**

Learners learn best when they are actively involved in the learning process. They should participate actively and contribute to a lesson. Each learner is an individual with their own needs, pace of learning, experience and abilities.

Teaching strategies must therefore be varied but flexible within a well-structured sequence of lessons. Learner-centred education gives the teacher the responsibility of facilitating and guiding so that learning takes place.

### **Role of the teacher**

In the competence-based syllabus, the teacher is a facilitator, organiser, advisor and a conflict solver. The following are specific duties of the teacher using a competence-based approach:

1. He or she is a facilitator. His or her role is to provide opportunities for learners to meet problems that interest and challenge them and that, with appropriate effort, they can solve. This requires an elaborated preparation to plan the activities, where they will be carried out and the required assistance.
2. He or she is an organizer. His or her role is to organise the learners in the classroom or outside and engage them in participatory and interactive methods through the learning processes as individuals, in pairs or in groups. To ensure that the learning is personalized, active and participative, the teacher must identify the needs of the learners, the nature of the learning to be done and the means to shape learning experiences accordingly
3. He or she is an advisor. He or she provides counselling and guidance for learners in need. He or she comforts and encourages learners by valuing their contributions in the class activities.
4. He or she is a conflict-solver. Most of the activities are competence-based are performed in groups. The members of a group may have problems such as the distribution of tasks. They should find useful and constructive the intervention of the teacher as a unifying element.

5. He or she is ethical and preaches by examples, by being impartial, by being a role-model, by caring for individual needs, especially for slow learners and learners with varied impairments, through a special assistance by providing remedial activities or reinforcement activities.

## **Role of the learner**

In the competence-based syllabus, the learner is the principal actor of his or her education. He or she is not an empty bottle to fill. Taking into account the initial capacities and abilities of the learner, the syllabus lists under each unit, the activities which the learners can engage in during the learning process.

The teaching-learning processes will be tailored towards creating a learner-friendly environment based on the capabilities, needs, experiences and interests. The following are some of the roles or the expectations from the learners:

1. Learners construct the knowledge either individually or in groups in an active way. From the learning theory, learners move in their understanding from concrete through pictorial to abstract. Therefore, the opportunities should be given to learners to manipulate concrete objects and to use models.
2. Learners are encouraged to use hand-held calculators. This stimulates learning of mathematics and acknowledging its importance as it is used both in jobs and in scientific applications.
3. Learners work on one competency at a time in form of concrete units with specific learning objectives broken down into knowledge, skills and attitude.
4. Learners are encouraged to do research and present their findings through group work activities.
5. A learner is cooperative; learners work in heterogeneous groups to increase tolerance and understanding.
6. Learners are responsible for their own participation and ensure the effectiveness of their work.
7. Help is sought from within the group and the teacher is asked for help only when the whole group agrees to ask a question
8. The learners who learn at a faster pace do **not** do the task alone as the others merely sign it off.
9. Teacher ensures the effective contribution of each learner, through clear explanation and argumentation to improve the English literacy and to

develop a sense of responsibility and to increase the self-confidence, the public speech ability, etc.

## Competence-based assessment

Competence-based assessment is an assessment process in which a learner is confronted with a complex situation relevant to his or her everyday life and is tasked to look for a solution by applying what has been learned (knowledge, skills, competences and attitudes). Evidence of learning is then collected and used as the basis on which judgments are made concerning learner's progress against fixed performance criteria.

### When to assess

During the teaching and learning of mathematics, assessment should be clearly visible in lesson, unit, term and yearly plans.

- **Before learning (diagnostic):** At the beginning of a new section of work; to find out what learners already know and can do, and to check whether the learners are at the same level.  
Use: (i) Probing questions when a unit/topic is being introduced for the first time. Example: *Who can tell us what you understand by trigonometry?*  
(ii) Probing questions about the previous lesson (*From what was discussed yesterday*). Example: *Who can remind us what the main trigonometric ratios?*
- **During learning (formative/continuous):** When learners appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving learners support and feedback.
- **After learning (summative):** At the end of a section of work or a learning unit, the teacher has to assess after the learning. This is also known as *Assessment of Learning* to establish and record overall progress of learners towards full achievement.

### What to assess

- (a) Knowledge and understanding

The assessment should focus on correctness of answers, coherence of ideas, logical reasoning and understanding. The teacher should use high order thinking verbs like: identify, explain, indicate, discuss, predict, estimate, judge, etc.

(b) Assessing practical skills

Learners should show evidence of the ability to perform and accomplish a given task through aptitude and or use practical tests and evaluation of the final outcome of learning. The assessment should focus on accuracy, quality product, correctness, speed and efficiency, coherence.

(c) Assessing attitude and values

The assessment should focus on the learner's approach to a situation, appreciation of the task given, impression of a situation, manipulation, reasoning, persistence, tolerance.

(d) Assessing generic competences:

Judgment capacity using verbs like: arrange, develop, subdivide, point out, design, produce, organise, develop, integrate, apply, discover, survey, produce, etc. depending on the generic competence assessed.

The knowledge, skills, attitudes and generic competences are not assessed independently of each another. It is important to set tasks which give evidence for the key aspects of topic or unit.

The lesson, unit or subject concept is the major focus but the style of assessing especially through questioning dictates all components being assessed. One question can cover the concept, all or part of generic competences, attitude and practical skills.

One must ensure that the verbs used in the formulation of questions do not require memorization or recall answers only but also test skills and attitudes as well as generic competences as stated in the syllabus.

## Instruments of assessment

Instruments of assessment are the tools used to establish whether learning is being (or has been) achieved. These can be used before, during or after learning. The teacher can select the appropriate instruments to use in assessment. The following are some of the instruments that can be used in assessment of mathematics.

1. **Observation:** This is where the teacher gathers information by watching learners interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it can be used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.

## 2. Questioning

- a. **Oral questioning:** a process which requires a learner to respond verbally to questions
- b. **Class exercise:** tasks that are given during the learning and teaching process
- c. **Quiz:** short and informal questions usually asked during a lesson
- d. **Homework and assignments:** tasks assigned to learners to be completed outside the classroom. These may include writing, problems to be solved, a school project to be built (display), or other skills to be practised.

### Good question items include:

- clear, simple and straightforward
- short and precise
- free of bias
- readable
- original
- indicate marks for each
- follow order of difficulty
- contain a variety of verbs.

3. **Portfolio:** Learner portfolios are a collection of evidence, prepared by the learner and evaluated by the teacher to demonstrate mastery, comprehension, application and synthesis of a given set of concepts.
4. **Project work:** A product which requires a learner to plan, carry out, and make a project presentation which is then assessed by the teacher or by peers
5. **Interview:** A process where a learner is expected to respond to questions concerning his or her learning
6. **Role play (enactment):** A performance which requires a learner to act out roles of other people in society in order to learn from their experiences. For example learners may dramatize the banking process showing the roles of various people and documents used
7. **Debate:** a performance which puts one learner, or team of learners, against another learner, or team of learners, to logically argue issues.

## **How to plan for assessment**

The process of planning an assessment involves a number of steps depending on the type of assessment.

- I Design tasks, set criteria, design rubrics and prepare appropriate questions beforehand, and decide how and when they are to be administered
- II Choose an appropriate method and technique to use either by observing, having dialogue and interactions with learners, organizing practical investigations, presentations and discussions, questioning orally or through paper and pen by giving quizzes, exercises or tests
- III Make provision for the learners' roles in self-assessment and peer assessment
- IV Develop assessment schemes for written work and products such as artwork, case studies, reports or project work presentations.

## **Developing competences**

### **The role of teachers in developing competences**

Teachers are not required to teach the way they were taught. They must embrace the new approaches with the aim of developing competences in the learners. This requires them to shift from teacher-centred to learner-centred methods. The following are important points to consider while implementing the competence-based curriculum:

- From the syllabus units, the teacher identifies different competences to be developed by the learners which are fostered by engaging learners through inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- The teacher focuses on observation of evidence on what learners can do and then identifies any difficulties encountered by them so that appropriate strategies can be developed for those with special needs (slow learners, learners with disabilities, talented and gifted learners).
- The teacher should take into account different cross-cutting issues and integrate them in the learning activities where applicable.
- The teacher should encourage individual, peer and group evaluation of the work done in the classroom. They must also use appropriate competence-based assessment approaches and methods.

- The teacher is a facilitator and a guide in the learning process. They must provide supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- The teacher is an advisor and provides guidance and counselling for learners. They support and comfort learners by valuing their contributions in the class activities.
- The teacher acts as a parent and has to ensure discipline, follow up learners' behaviour, and communicate with parents about the learners' performance at school.

### Note

#### The teacher should remember that during the teaching and learning process:

- Learners communicate and share relevant information with other learners through presentation, discussions, group work and other learner centred activities (role play, case studies, project work, research and investigation)
- Learners are active participants and take some responsibility for their own learning
- Learners develop knowledge and skills in active ways
- Learners carry out research and investigation, consulting print and online documents and resourceful people and present their findings
- During the assigned tasks, learners ensure the effective contribution of each group member, through clear explanation and arguments critical thinking, responsibility and confidence in public speaking

### Some strategies to develop the competence according to domains of learning

Domain of learning	What teachers can do	Examples of learning activities
Psychomotor domain	- Allow the learner to practise for a while, and then ask for a demonstration of the skill	- Observe a skill and attempt to repeat it, or see a finished product and attempt to replicate it (imitate)

Psychomotor domain	<ul style="list-style-type: none"> <li>- Set up models or create a simulation exercise in the practical laboratory where learners can have repeated practice of skills with peers and or teacher supervision</li> <li>- Arrange for sufficient practical experiences requiring skill performance under direct supervision</li> <li>- Create a valid and reliable assessment tool for use in determining competence in skill demonstration</li> </ul>	<ul style="list-style-type: none"> <li>- Produce the product by following general instructions rather than observation (manipulate)</li> <li>- Performing individual or group practical work to demonstrate particular skills under direct supervision of the teacher (manipulate)</li> <li>- Making models or designs related to the broad competence or specific learning outcomes with accuracy (precision)</li> <li>- Make accurate observations and draw appropriate conclusions from practical demonstration of a task by a teacher or fellow learners (manipulating with precision)</li> <li>- Studying situations through field visits and case studies</li> <li>- Undertaking project work with guidance from the teacher but with minimum supervision</li> </ul>
Cognitive domain	<ul style="list-style-type: none"> <li>- Develop case studies requiring discovery or problem-based learning to determine the most appropriate evidenced based example</li> <li>- Structure debates that require the learner to provide their reasons for their responses</li> <li>- Avoid the temptation to answer every learner's question, especially when the learner knows or should know the answer</li> <li>- Set self-study modules with suggested learning activities that the learners can complete on their</li> </ul>	<ul style="list-style-type: none"> <li>- Self-directed reading and completion of suggested activities that will add to learners' knowledge and experience base</li> <li>- Becoming active participants in and taking responsibility for their own learning</li> <li>- Discovering the best solution to a given need or problem in both theoretical and practical work</li> <li>- Retrieving and retaining knowledge and applying it in practice</li> <li>- Learning activities structured for groups of learners working together and self-directed</li> </ul>

	<p>own prior to interaction with fellow learners and teachers</p> <ul style="list-style-type: none"> <li>- Provide ample time for discussion and clarification of concepts to be learned</li> <li>- Help learners use their own knowledge and ideas to find possible solutions to situations.</li> <li>- Guide learners to discover how to proceed or act through higher order questioning (Socratic questioning)</li> </ul>	<p>using the world wide web or internet and intranet for resources related to topics being learned</p> <ul style="list-style-type: none"> <li>- Preparing for discussions and debates</li> </ul>
<p>Affective domain (Attitudes and values)</p>	<ul style="list-style-type: none"> <li>- Create an environment for learners to do exercises on positive and negative personal or peer values</li> <li>- Provide a framework for a written analysis of attitudes, values and behaviour</li> <li>- Structure opportunities for role play requiring recognition of differing values and behaviour</li> <li>- Join with learner group to discuss different values and beliefs especially those related to learning styles and interpersonal relationship.</li> <li>- Create a valid and reliable assessment tool for use in determining positive attitude demonstration</li> </ul>	<ul style="list-style-type: none"> <li>- Respond willingly and positively when asked or directed to do something</li> <li>- Comply with given expectations by attending or reacting to stimuli in agreeable manner</li> <li>- Display behaviour consistent with attitudes and behaviour acceptable in different situations</li> <li>- Listen to others and pay attention to any guidance and advice given by mentors</li> <li>- Reflect on how personal values promote or inhibit their ability to learn better and to fit in the society</li> <li>- Identifying role models in the school system and in the community and listing the qualities they admire.</li> </ul>

## Techniques to develop competences

The teacher can use the following techniques while teaching mathematics to support development of competences:

Techniques/ Strategies	Description
Roundtable	This is a form of cooperative learning. A question is posed by the teacher to groups of learners. Each person in the group writes one answer on a paper and passes it to the next team member. The group looks at each answer and decides which one to present to the class. Each group shares or presents their answer to the entire class. The suggestions are discussed by the class and conclusions drawn.
Questions in corners	The teacher places questions in different corners of the classroom. Groups of 3-6 learners move from corner to corner as per signal given by the teacher. They discuss and write an answer to each question taking into account answers already written by previous groups. The use of different coloured markers for each group helps to see what each group wrote for each question. Ideas for each question are discussed in plenary to come up with some conclusions at the end.
Outdoor activities  Field visits	In field visits, learners go outside the classroom to observe specific organisms or phenomena, or to hear information from experts. Before the visit the teacher and learners: <ul style="list-style-type: none"> <li>- agree on aims and objectives</li> <li>- gather relevant information prior to visit</li> <li>- brainstorm on key questions and share responsibilities</li> <li>- discuss materials needed and other logistical issues</li> <li>- discuss and agree on accepted behaviours during the visit</li> </ul> After the visit: <ul style="list-style-type: none"> <li>- de-brief and discussion of what was learned and observed</li> <li>- evaluation of all aspects of visit</li> <li>- reports, presentations prepared by learners</li> </ul>
Project work	Learners in groups or individually, are engaged in a self-directed work for an extended period of time to investigate and respond to a complex question, problem, or challenge. The work is presented to classmates and other people beyond the school. Projects are based on real-world problems that capture learners' interest. This technique develops higher order thinking as the learners acquire and apply new knowledge in a problem-solving context.

	<p>The teacher plays the role of facilitator by:</p> <ul style="list-style-type: none"> <li>- working with learners to frame worthwhile questions</li> <li>- setting relevant and meaningful tasks</li> <li>- availing resources needed</li> <li>- coaching both knowledge and skills development and social skills,</li> <li>- assessing carefully what learners produced based on defined criteria</li> </ul>
Group work	<p>This is a form of peer/cooperative/ collaborative learning that values the learner-learner interaction. It is mutually beneficial and involves the sharing of knowledge, ideas and experience between learners. It offers learners opportunity to learn from each other.</p> <p>To be effective, teams should be heterogeneous in terms of ability levels, made of 3-4 learners in most tasks. Team members are assigned specific roles which are rotated. For elaborated work, assessment should be twofold: based on both the collective and individual work</p>
Role play	<p>The role play is a special kind of case study in which there is an explicit situation established with learners playing specific roles, spontaneously saying and doing what they understand their “character” would do, in that situation. The case study differs from the role play because in the case study, learners read about situations and characters; in the role play, they find themselves what to say, how to play and which material to use.</p>
Case study	<p>Case study as a learning technique is a story either based on real events, or from a construction of events which could reasonably take place. It involves issues or conflicts which need to be resolved. The information contained in a case study can be complex or simple.</p> <p>The teacher presents a problem situation and indicates how to proceed</p>
Brainstorming	<p>It is a technique used for creative exploration of options/ solutions in an environment free of criticism. It encourages creativity and a large number of ideas.</p> <p>Among ground rules there are: active participation by all members; no discussions, criticisms, compliments or other comments during the brainstorming stage. The teacher starts by reviewing the rules, sets a time limit; states and explains the question; collects and displays ideas; eliminates duplications and guides learners to draw a conclusion.</p>

<p><i>A learning centre/ corner</i></p>	<p>It is a space set aside in the classroom that allows easy access to a variety of learning materials in an interesting and productive manner. Learners can work by themselves or with others in self-directed activities on a content related to the curriculum or not.</p> <p>These centres allow learners to deepen their understanding of subjects, apply their learning in a stimulating learning environment and engage in meaningful discoveries that match their individual interests. They provide learners with hands-on experiences they can pursue at their own pace and level of curiosity.</p>
<p>Games/play</p>	<p>Games are used to help learners to learn faster and better, and in enjoyable manner. Games and plays help to create a classroom experience that actively engages learners. They develop communication and other important skills such as social skills, critical thinking, problem-solving, numeracy and literacy skills in different subjects.</p>
<p>Research work</p>	<p>Each learner or group of learners is given a research topic. They have to gather information or ask experienced people and then the results are presented and discussed in class.</p>
<p>Practical work</p>	<p>Individually or in teams, learners are assigned practical tasks. To be effective, the task needs: a clear purpose with strong links and relevance to the curriculum; quality materials; learners' engagement; time for preparation and carrying out the work; support from the teacher or other experts. Such activities encourage deeper understanding of phenomena; develop skills such as observation, practical work, planning, reporting, etc.</p>

## Resources

Learning and teaching materials and resources refer to a variety of educational materials that teachers and learners use in the classroom to support specific learning objectives. The learner-centered approach in the mathematics syllabus delivery emphasise the need to use a variety of teaching learning resources including those improvised or collected by the teacher and the learners from the surrounding environment.

### Identifying resources

Before planning and delivering a lesson, resources should be identified at school or in the surrounding environment according to the lesson. Examples of resources a teacher can use in teaching and learning include:

- **Library:** Textbooks, dictionaries, reading books, flip charts
- **ICT equipment:** laptop and - desktop computers, projectors, mobiles devices, IWB, TV, radios, smart board, smart phones, mobile phones, CD-ROMs, flash discs and digital cameras.
- **Digital (electronic) materials:** audio, video, interactive, simulators, animations, digital images, Internet content, software, PPT and DOC.
- **Real objects:** sticks, bottle tops, clothing, food packaging and plastic bottles, etc.
- **Materials from the environment:** soil, vegetables, animals, home or domestic objects
- **Human resources:** learners and people in the local community, parents, local leaders, role models, etc.

From the available resources, a teacher has to select materials considered to be the best and most suitable for the particular learning activity, and reject what is inappropriate or unsuitable.

**Note:**

Teachers are encouraged to be creative and innovative and use the available resources in their environment, be it in the classroom, at school, and the community

## **Inclusiveness in the class**

Inclusion is based on the right of all learners learning together for a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity. All learners have the right to access education regardless of their different impairments and this implies that all citizens benefit the same menu of education programs.

## **Special educational needs**

Special educational needs are learning difficulties or a disabilities which make it harder for children to learn in the same way as their peers of the same age.

## **How to deal with special needs education learners**

Learners with special needs education challenges fall under many categories and a teacher needs to know these challenges, their signs and ways of helping these learners so as to attain education also.

### **(a) Physical challenges**

This can be seen when a learner has physical (body) deformation. Such learners find difficulty in walking and some have sitting and standing posture problems. The teacher can help these learners by:

- using other learners to help them where possible.
- increasing space between desks to accommodate crutches, wheelchairs, etc.
- adopting activities that suit the learners.
- modifying certain aspects of the classroom and school environment such as corridors and staircases.

### **(b) Visual impairment**

Learners with these problems always show different signs. These are visual complaints, reading problems like reading when the book is too far or too near to the eyes, rapid eye movement, squinting (of eyes), watery and reddish eyes and tripping over things. These learners can be helped by:

- giving them the front seats in class
- using larger font materials
- providing Braille for reading and writing
- allowing more time to internalize concepts and complete tasks
- use tangible learning aids or those that can be manipulated.

### **(c) Hearing impairment**

A learner with these problems has listening and speaking problems, is always inattentive in class, experiences pain in ears, has inflamed or running ears, does not respond to sound, slow in starting an interaction, is lonely and always speaks loudly. The teacher can help this learner by:

- Speaking loudly when teaching,
- using visual learning materials and sign language
- doing a lot of writing when teaching mostly
- using oral and written assessment

### **(d) Mental challenges**

Learners with mental challenges exhibit the following symptoms. They are slow learners, show no interest in learning, do not follow instructions well, depend on imitating what others have done and have poor hand and eye coordination. A teacher can help such learners by doing the following.

- Providing opportunities to the learners to experience success and achievement.
- Giving tasks equal to their interest and ability
- Increase use of visual aids
- Giving praise when correct response is given
- Making instructions simple
- Communicating simple concepts first
- Breaking tasks into smaller units
- Teaching one skill or concept at a time
- Using drill and repetition
- Starting instructions from the simple to the complex ones.

#### **(e) Speech problems**

Learners with speech problems may have problems such as stammering, speaking too fast or too slowly, poor at sentence construction, not asking questions in class, too shy to speak in class, sound distortion and poor intonation of words. A teacher can help such a learner by:

- practising speech lessons in class
- using short phrasal sentences
- using direct language when talking to such a learner
- using visual aids to illustrate concept
- demonstrating to learners, besides telling them what and how to do things.

#### **(f) Gifted and talented learners**

Learners who are highly gifted and talented in learning are always identified because they ask questions in class, find no difficulty in doing classwork, learn easily and fast, remember easily what they have learnt before, research, are very good in speaking and self-expression as well as fluency and they ask difficult thought-provoking questions. A teacher can help such a learner by doing the following.

- Giving them enough work.
- Giving them work to research about so as to keep them occupied.
- Encouraging such learners to read ahead of the teacher's work in class.
- Give them responsibilities that will keep them busy like leading their groups.

- Facilitate creativity and exploration of the environment.
- Be there promptly when they need you.
- Providing accelerated, investigative and enriched exercises.

### **(g) Indisciplined learners**

At times some learners have disruptive behaviour. A teacher has the responsibility to help them attain education.

Such learners usually exhibit the following: aggressive in class, cruel and bullies others, tend to depend on others, not willing to participate in class, unhappy, rude and lacks respect, sluggish, lazy, absenteeism, inability to create and maintain relationship and are quiet and inactive. A teacher can help such learners by:

- guiding and counselling them
- setting class rules and regulations
- occupying such learners with work
- providing a favourable class environment
- motivating them positively and discouraging what is not good for them
- teaching them the expected behaviours and manners of a learner
- being firm in handling issues and make it clear
- cultivating a friendly atmosphere

### **Cross-cutting issues**

The competence-based curriculum reflects the significance of connections between different subject areas, integrating them across years and cycles. Cross-cutting issues are integrated across learning areas appropriately. They are all important for learners to learn about, but they are not confined to one subject.

Cross cutting issues are not stand-alone subjects. They are issues which cut across the entire curriculum. There are eight (8) cross cutting issues:

- **Peace and values education:** how education can simultaneously cultivate values and attitudes which will encourage individual and social action for building more peaceful selves, families, communities, societies, nations and ultimately a more peaceful world.
- **Genocide studies:** helps learners to comprehend the role of every individual in ensuring that genocide never happens again.
- **Gender education:** The gender of a person is biologically determined, whereas ways of being a man or a woman are learned: they are

constructed, reinforced, maintained and reconstructed over time through social and cultural practices.

- **Inclusive education:** learning needs are to be considered and accommodated for when teaching each learning expectation.
- **Comprehensive sexuality education:** equips children, adolescents and young people with the knowledge, skills and values in culturally and gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, promote and sustain risk-reducing behaviour.
- **Financial education:** build a strong foundation among the learners for responsible money management by developing good planning and saving habits and prepare them for life, such as managing their own finances.
- **Environment and sustainability:** It is important to realize that humans enjoy a unique position in nature due to their exceptional ability to influence and mold the environment.
- **Standardization culture:** prepares for their responsibilities as adults to contribute to, for example, health improvement, economic growth, industrialization, trade and general welfare

In Mathematics, cross-cutting issues have been integrated into the units. So, it is the role of the teacher to include them in his/her teaching.

## Professional documents

A teacher needs three main professional documents for instruction in class. These are:

- Scheme of work
- Lesson plan
- Record of work covered

These documents are meant to ensure effective teaching and easy management of the teaching and learning process. A teacher must prepare and use these professional documents while teaching. The use of these documents promotes the following.

- Proper and prior planning of the teaching process.
- Effective and objective teaching.
- Time management.
- Uniformity in the teaching process.
- Easy takeover of the subject by incoming teachers.

## **(a) Scheme of work**

This is a laid out guideline that defines the structure and content of the work to be taught in a term. It maps out clearly how resources such as books, equipment, time and class activities such as lectures, group work and practical work will be used to ensure that learning aims and objectives are met successfully.

A well planned scheme of work will guide the teacher in teaching at the right pace and to cover the planned work as scheduled. Each unit of the syllabus is studied carefully before preparing a scheme of work.

Equally important is the careful allocation of sufficient time for each topic. This is to ensure that all the learning objectives of a topic are adequately achieved. Any likely interference to lessons within the term should be taken into account when allocating time to the units in the schemes of work. Such interruptions could include public holidays, inter-house or zone sports competitions, mid-term break, and prize-giving day and examination days. The school calendar should be given to the teachers before the opening of each term.

# Template of a sample scheme of work

Academic year: .....

Term: .....

School: .....

Subject: .....

Teacher's name: .....

Class: .....

Dates and number of lessons (periods) in a week	Units + Key unit competencies	Lessons + Evaluation	Learning objectives	Teaching methods and techniques + Evaluation procedures	Resources and References	Observations
From January 11 (Monday) to January 15 (Friday) <b>3 periods</b>	Unit 1 <b>Unit competence</b> Use trigonometric circles and identities to determine trigonometric ratios and apply them to solve related problems.	Lessons 1, 2 and 3	Knowledge and understanding: <ul style="list-style-type: none"> <li>Convert radians to degrees and vice versa.</li> </ul> Skills: <ul style="list-style-type: none"> <li>Measure angles and verify the radian measure.</li> </ul>	<ul style="list-style-type: none"> <li>Lecture</li> <li>Discussions</li> <li>Question and answer sessions</li> <li>Written exercises</li> </ul>	<ul style="list-style-type: none"> <li>Achievers Mathematics Senior 4 Student's Book pages 1 to 5.</li> </ul>	
From January 18 (Monday) to January 22 (Friday) <b>3 periods</b>		Lessons 4, 5 and 6	Knowledge and understanding: <ul style="list-style-type: none"> <li>Define sine, cosine and tangent of any angle.</li> </ul> Skills: <ul style="list-style-type: none"> <li>Use the unit circle to relate values of any angle to the value of positive acute angles.</li> </ul> Attitudes and values: <ul style="list-style-type: none"> <li>Appreciate the relationship between trigonometric values for different angles.</li> </ul>	<ul style="list-style-type: none"> <li>Lecture</li> <li>Discussions</li> <li>Question and answer sessions</li> <li>Written exercises</li> </ul>	<ul style="list-style-type: none"> <li>Achievers Mathematics Senior 4 Student's Book pages 5 to 7.</li> </ul>	
		Summative Evaluation 1		Do the tasks in the Student's Book and teacher to give more exercises found in the Teacher's Guide.		

## **(b) Lesson plan**

A lesson plan is a teacher's detailed description of what he or she is going to teach. A lesson plan is an extract from a scheme of work. A daily lesson plan should be made by the teacher to guide him/her in teaching.

The teacher will find his or her work easier if she or he goes to class well prepared with the lesson content organised in a logical manner. Good preparation gives the teacher confidence needed to teach well.

## Template of a competence-based lesson plan

School name: .....

Teacher's name: .....

Term	Date ..... / ..... / 20.....	Subject .....	Class .....	Unit N° ....	Lesson N° ... of ....	Duration ...	Class size ....
Type of Special Educational Needs and number of learners							
Topic area:							
Sub-topic area:							
Unit title							
Key Unit competence:							
Title of the lesson							
Instructional objective							
<p>Note:</p> <p>Set by the teacher based on learning objective from the scheme of work</p> <p>Needs to be inclusive to reflect needs of whole class</p> <p>Focuses on the 5 elements of an instructional objective and includes the three categories of learning objectives (K &amp; U; S; A &amp; V) where possible. It is not always possible to cover the three categories of learning objectives in 40 minutes.</p>							
Plan for this Class (location: in / outside)							
Learning materials (for all learners)							

References		Generic competences and cross cutting issues to be addressed	
Timing for each step	Description of teaching and learning activity		
	Teacher activities	Learner activities	
Introduction ...min			
Development of the lesson ...min			
Conclusion ... min			
Teacher self-evaluation			

### **(c) Record of work covered**

This is a departmental subject record book which shows how much work has been covered in each lesson and for each week. It is very handy when, for some reason, the class has to be passed on to another teacher. The new teacher will, in this case, know the extent of coverage of the topic.

The record of work also helps the teacher to know where to start when preparing for the next lesson; more so after a long break (like a weekend). It is unprofessional for the teacher to ask the learners to remind him for her where he or she ended the previous lesson.

Columns which are common to those in the schemes of work are used, but again, the format could vary from one teacher or school to another. The major entries should include:

1. Week
2. Lesson
3. Main topic
4. Lesson topic/sub-topic
5. Content
6. Remarks

### **Assessment approaches**

Assessment refers to the process of evaluating the teaching and learning process by collecting and interpreting evidence of an individual learner's progress in learning and to make a judgement about a learner's achievements measured against defined standards.

Assessment is an integral part of the teaching-learning process. In the new competence-based curriculum, assessment must also be competence-based, whereby a learner is given a complex situation related to their everyday life and asked to try to overcome the situation by applying what they have learnt. Assessment should be organised in class as individual or group work.

### **Objectives of assessment**

- To find out how much the learners have been able to master in your lesson.
- To find out whether the objectives have been achieved or not.
- To be able to plan well for the next lesson.
- To revise the covered work with the learners.

- To determine whether the main concepts have been mastered by the learners.
- To be able to choose learners who qualify to move into the next learning level.

## **Types of assessment**

### ***Formative and continuous assessment***

This involves using formal and informal methods to ascertain whether learning is taking place. When planning a lesson, establish the criteria for performance and behaviour changes at the beginning of each unit. Then at the end of each unit, ensure that all the learners have acquired the stated key unit competences based on the stated criteria before beginning the next unit.

As a teacher, assess how well each learner masters both the subject and the generic competences described in the syllabus.

This will give the teacher a clear picture of the all round progress of the learners. The teacher will use one or a combination of the following.

- Observation method
- Pen and paper
- Oral questioning

### *Forms of formative assessment*

#### *(a) Observation*

A teacher should study and learn more about the learner so that he or she is able to know what the learner knows or does not know. The teacher should record useful data about the learner. This will help you as a teacher to know where the learner may need help.

#### *(b) Interviews*

A teacher can set tasks or problems to determine a student's understanding about a concept or set of related ideas. This kind of assessment provides feedback that is especially useful to teachers.

#### *(c) Peer assessment/self assessment*

This kind of assessment interests learners as it creates a learning community within the classroom. When learners are involved in goal-setting, self evaluation becomes a logical step in the learning process. The teacher can evaluate the self-assessment and peer assessments and identify students' strength and weaknesses.

#### *(d) Discussion*

Discussion in class can tell you as a teacher much about student learning and understanding of the concepts. Initiate the discussion by giving learners an open-ended question. This helps to build knowledge and develop critical and creative thinking skills.

It also develops social skills, communication and leadership skills. The teacher can assess learner understanding by listening to the learners' responses and taking notes.

#### **Summative assessment**

Summative assessment is used to record a judgement of competence or performance of the learner. This assessment gives the teacher a picture of a learner's progress at any specific moment. The main objective of this assessment is to determine whether learning objectives have been achieved and use the results for grading learners.

This assessment is used for deciding on progression or selection into the next level of learning and awarding of certificates.

### **Record keeping**

This refers to the gathering of facts and evidence from assessment instruments and using them to judge the learner's performance by assigning an indicator against a set criteria or standard.

As regards a class evaluation, as a teacher you will always keep a record of every learner's performance so as to see a learner's progress in class and devise means of improving that learner's performance and competency acquisition.

### **Teaching resources**

These are materials used to elaborate more about what you are teaching and to make teaching flexible. Materials and things in the environment such as animals, plants, hospitals, rivers, swamps, factories, game parks and churches are used. Resource persons are also used. All these are referred to as teaching resources. They should be used effectively while teaching.

#### ***Real objects***

Realia (real objects) can be used in class. They can be brought to class or used out of class. They are mostly the things found in the local environment. If they are things class, the teacher is expected to take learners out to see, and touch

outside. This will help the learners to digest what they have learnt. Examples of realia include plants, animals, clothes, food, building materials and weather instruments.

### ***Pictures and charts***

These are drawn and used in line with the lesson to be taught. They help learners to get important facts in the lesson in most cases where real objects cannot be used or are not available. Pictures and charts should be enough for the learners in class and should be put in a place where all learners can see them clearly.

### ***Resource centres***

These are places where important information can be got from books and other materials and used in teaching and learning Mathematics. Each school should try and have a library. It should contain educational materials such as books, magazines, newspapers and pictures. The learners can get information from these materials on their own when they visit the library. They can also be taken on field visits to the other resource centres that cannot be found within the school.

### ***Textbooks***

Textbooks are the most commonly used teaching and learning materials. The teacher needs to use them together with other materials or resources to make the teaching-learning process interesting. It will also help to capture the attention of learners and prevent monotony. The teacher needs to be creative and use other teaching learning materials together with the textbooks.

### ***Audio-visual devices***

Audio-visuals are those things that possess both sound and visual components. They include televisions, videos, computers, films and power point presentations. Here the teacher uses recorded work to put more emphasis on what she or he is teaching while learners are watching whatever she or he tells them. It makes learning interesting, fun and practical.

## **Mathematical approaches**

- 1. Facts:** Symbols of mathematics. These are learnt through memorization, drill, practice, timed tests, games and concerts. Students are said to have learnt facts when they can state them.
- 2. Skills:** Mathematical skills are operations and procedures. Many skills can be specified by sets of rules and instructions or by ordered sequence

of specific procedures called algorithms. Skills are learnt through demonstrations, drill, practice and group activities. By applying a skill in different situations (solving different problems requiring the skill), the student has acquired the skill.

- 3. Concepts:** A concept in mathematics is an abstract idea which enables people to classify objects or events and to specify whether the objects and events are examples or non-examples of the abstract idea.

Sets, sub-sets, inequality, exponent, triangle, are examples of concepts. Concepts can be learned through direct observation or through definitions. By direct observation and experimentation, they learn to classify plane objects into sets of triangles, circles or squares.

A concept is learned by hearing, seeing, handling, discussing or thinking about a variety of examples and non-examples of the concept. It is also learned by contrasting the examples and non-examples.

Students are said to have learned concepts when they can distinguish examples from non-examples of the concept.

- 4. Principles:** Sequences of concepts together with relationships among these concepts. The Pythagoras's Theorem ( $a^2 + b^2 = c^2$ ) is an example of a principle. Principles are learned through processes of scientific inquiry, group discussion, use of problem solving strategies and demonstrations.

A student is said to have learned a principle when she or he can identify the concepts included in the principle, put the concepts in their correct relation to one another, and apply the principle to a particular situation.

Teachers should develop testing and observation techniques to assist in recognising the learners' viewpoints of the concepts and principles taught.

## Content map for Student's Book 4

	Unit 1: Fundamentals of trigonometry	Unit 2: Propositional and predicate logic	Unit 3: Binary operations
<b>Number of periods</b>	26 + homework	14 + homework	14 + homework
<b>Classroom organisation</b>	Whole class orientation; then group work.	Whole class orientation; individual work and then working in groups.	Whole class orientation; then working in groups.
<b>Equipment required</b>	Geometrical instruments, ruler, T-square, compass, protractors, graph papers, digital instruments such as calculators	Manila papers, markers, rulers	Digital instruments such as calculators, counters
<b>Activities</b>	<ul style="list-style-type: none"> <li>Defining sine, cosine, tangent, cosecant, secant and cotangent of any angle; know special values (<math>30^\circ</math>, <math>40^\circ</math>, <math>60^\circ</math>, <math>30^\circ</math>, <math>45^\circ</math>, <math>60^\circ</math>)</li> <li>Converting radians to degrees and vice versa</li> <li>Differentiating between complementary angles, supplementary angles and co-terminal angles.</li> <li>Representing graphically sine, cosine and tangent functions and, together with the unit circle, use to relate values of any angle to the value of a positive acute angle.</li> <li>Using trigonometry, including the sine and cosine rules, to solve problems involving triangles.</li> </ul>	<ul style="list-style-type: none"> <li>Distinguish between statement and proposition</li> <li>Convert into logical formula composite propositions and vice versa</li> <li>Draw the truth table of a composite proposition</li> <li>Recognise the most often used tautologies (example: De Morgan's Laws)</li> <li>Using mathematical logic to infer conclusion from given proposition</li> <li>Showing that a given logic statement is tautology or a contradiction</li> </ul>	<ul style="list-style-type: none"> <li>Defining a group, a ring, an integral domain and a field</li> <li>Demonstrating that a set ( or is not) a group, a ring or a field under given operations</li> <li>Demonstrating that a subset of a group is (or is not) a sub group</li> <li>Determining the properties of a given binary operation</li> <li>Formulating, using adequate symbols, a property of a binary operation and its negation</li> <li>Constructing the Cayley table of orders 2, 3, 4</li> <li>Discovering a mistake in an incorrect operation</li> </ul>

<p><b>Competences to be practised</b></p>	<ul style="list-style-type: none"> <li>• Communication</li> <li>• Problem-solving</li> <li>• Research</li> <li>• Cooperation</li> <li>• Critical thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Communication</li> <li>• Research</li> <li>• Cooperation</li> <li>• Problem-solving</li> <li>• Critical thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Problem-solving</li> <li>• Cooperation</li> <li>• Communication</li> <li>• Lifelong learning</li> </ul>
<p><b>Attitudes and values</b></p>	<ul style="list-style-type: none"> <li>• Appreciate the relationship between the trigonometric values for different angles</li> <li>• Verify reasonableness of answers to exercises when solving problems.</li> </ul>	<ul style="list-style-type: none"> <li>• Judge situations accurately</li> <li>• Observe situations and make appropriate decisions</li> <li>• Appreciate and act with thoughtfulness: grasp and demonstrate carefulness.</li> <li>• Develop and show mutual respect</li> <li>• Demonstrate broadmindedness</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate the importance and the use of properties of binary operations</li> <li>• Show curiosity, patience, mutual respect and tolerance in the study of binary operations</li> </ul>

	Unit 4: Set $\mathbb{R}$ of real numbers		Unit 5: Linear equations and inequalities		Unit 6: Quadratic equation and inequalities	
<b>Number of periods</b>	24 + homework	12 + homework	18 + homework			
<b>Classroom organisation</b>	Whole class orientation; then group work.	Whole class orientation; individual work and then working in groups.	Whole class orientation; then group work.			
<b>Equipment required</b>	Graph papers, manila papers, digital technology instruments including calculators	Geometric instruments (ruler, T-square ), digital technology instruments including calculators.	Geometric instruments (ruler, T-square), digital technology instruments including calculators			
<b>Activities</b>	<ul style="list-style-type: none"> <li>• Matching a number and the set to which it belongs</li> <li>• Defining a power, an exponential, a radical, a logarithm, the absolute value of a real number.</li> <li>• Classifying numbers into naturals, integers, rational and irrationals</li> <li>• Determining the restrictions on the variables in rational and irrational expressions</li> <li>• Illustrating each property of a power, an exponential a radical, a logarithm, the absolute value of a real number</li> <li>• Using logarithms and exponentials to model simple problems about growth, decay, compound interest, magnitude of an earthquake, etc</li> <li>• Transforming a logarithmic expression to equivalent power or radical form and vice versa</li> <li>• Rewriting an expression containing “absolute value” using order relation</li> </ul>	<ul style="list-style-type: none"> <li>• Listing and clarifying the steps in modelling a problem by linear equations and inequalities</li> <li>• Solving parametric equations and inequalities in one unknown.</li> <li>• Solving simultaneous equations in two unknowns</li> </ul>	<ul style="list-style-type: none"> <li>• Defining a quadratic equation</li> <li>• Solving problems related to quadratic equations</li> <li>• Applying critical thinking by solving any situation related to quadratic equations</li> <li>• Determining the sign, sum and product of a quadratic equation</li> <li>• Discussing the parameter of a quadratic equation and a quadratic inequality</li> <li>• Explaining how to reduce Bi-quadratic equations to a quadratic equation and other equations of degree greater than 2</li> </ul>			

<p><b>Competences to be practised</b></p>	<ul style="list-style-type: none"> <li>• Communication</li> <li>• Problem-solving</li> <li>• Research</li> <li>• Cooperation</li> <li>• Critical thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Communication</li> <li>• Research</li> <li>• Cooperation</li> <li>• Problem-solving</li> <li>• Critical thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Problem-solving</li> <li>• Cooperation</li> <li>• Communication</li> <li>• Lifelong learning</li> </ul>
<p><b>Attitudes and values</b></p>	<ul style="list-style-type: none"> <li>• Appreciate the importance and the use of properties of operations on real numbers</li> <li>• Show curiosity for the study of operations on real numbers</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate, value and care for situations involving two linear equations and inequalities in daily life situations</li> <li>• Show curiosity about linear equations and linear inequalities</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate value and care for situations involving quadratic equations and quadratic inequalities in daily life situations.</li> <li>• Show curiosity about quadratic equations and quadratic inequalities.</li> </ul>

	<b>Unit 7: Polynomial, rational and irrational functions</b>	<b>Unit 8: Limits of polynomial, rational and irrational functions</b>	<b>Unit 9: Differentiation of polynomials, rational and irrational functions and their application</b>
<b>Number of periods</b>	14+ homework	14+ homework	21+ homework
<b>Classroom organisation</b>	Whole class orientation; then group work.	Whole class orientation; individual work and then working in groups.	Whole class orientation; then working in groups.
<b>Equipment required</b>	Pair of compasses; graph paper, ruler, digital technology materials such as calculators	Manila papers, graph papers, rulers, markers, calculators	Manila papers, graph papers, digital technology including calculators, ...
<b>Activities</b>	<ul style="list-style-type: none"> <li>Identifying a function as a rule and recognising rules that are not functions</li> <li>Determining the domain and range of a function</li> <li>Constructing composition of functions</li> <li>Finding out whether a function is even , odd , or neither</li> <li>Demonstrating an understanding of operations on polynomials, rational and irrational functions, and finding the composite of two functions.</li> <li>Performing operations on functions</li> <li>Applying different properties of functions to model and solve related problems in various practical contexts.</li> <li>Analysing, modelling and solving problems involving linear or quadratic functions and interpreting the results</li> </ul>	<ul style="list-style-type: none"> <li>Defining the concept of limit for real-valued functions of one real variable</li> <li>Evaluating the limit of a function and extending this concept to determine the asymptotes of the given function.</li> <li>Calculating limits of certain elementary functions</li> <li>Developing introductory calculus reasoning.</li> <li>Solving problems involving continuity.</li> <li>Applying informal methods to explore the concept of a limit including one-sided limits.</li> <li>Using the concepts of limits to determine the asymptotes to rational and polynomial functions</li> </ul>	<ul style="list-style-type: none"> <li>Evaluating derivatives of functions using the definition of derivative.</li> <li>Defining and evaluate from first principles the gradient at a point.</li> <li>Distinguishing between techniques of differentiation to use in an appropriate context.</li> <li>Using properties of derivatives to differentiate polynomial, rational and irrational functions</li> <li>Using first principles to determine the gradient of the tangent to a curve at a point.</li> <li>Applying the concepts and techniques of differentiation to model, analyse and solve rates or optimisation problems in different situations.</li> </ul>

<p><b>Competences to be practised</b></p>	<ul style="list-style-type: none"> <li>• Problem-solving</li> <li>• Cooperation</li> <li>• Communication</li> <li>• Lifelong learning</li> </ul>	<ul style="list-style-type: none"> <li>• Communication</li> <li>• Research</li> <li>• Cooperation</li> <li>• Problem-solving</li> <li>• Critical thinking</li> </ul>	<ul style="list-style-type: none"> <li>• Communication</li> <li>• Problem-solving</li> <li>• Research</li> <li>• Cooperation</li> <li>• Critical thinking</li> </ul>
<p><b>Attitudes and values</b></p>	<ul style="list-style-type: none"> <li>• Increase self-confidence and determination to appreciate and explain the importance of functions</li> <li>• Show concern on patience, mutual respect and tolerance</li> <li>• When solving problems about polynomial, rational and irrational functions</li> </ul>	<ul style="list-style-type: none"> <li>• Show concern on the importance, the use and determination of limits of functions</li> <li>• Appreciate the use of intermediate-value theorem</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate the use of gradient as a measure of rate of change</li> <li>• Appreciate the importance and use of differentiation in kinematics (velocity, acceleration )</li> <li>• Show concern on derivatives to help in understanding optimization problems</li> </ul>

	Unit 10: Vector space of real numbers	Unit 11: Concepts and operations on linear transformation in 2D	Unit 12: Matrices of and determinants of order 2
<b>Number of periods</b>	16+ homework	14+ homework	12+ homework
<b>Classroom organisation</b>	Whole class orientation; then group work.	Whole class orientation; individual work and then working in groups.	Whole class orientation; then working in groups.
<b>Equipment required</b>	Manila papers, graph papers, geometric instruments: ruler, T-square, protractors, computers	Manila papers, graph papers, geometric instruments (ruler, pair of compasses, T-square), digital technology instruments including calculators	Manila papers, markers
<b>Activities</b>	<ul style="list-style-type: none"> <li>Defining the scalar product of two vectors</li> <li>Determining the magnitude of a vector and angle between two vectors</li> <li>Calculating the scalar product of two vectors</li> <li>Analysing a vector in terms of size.</li> <li>Calculating the angle between two vectors</li> </ul>	<ul style="list-style-type: none"> <li>Defining and distinguishing between linear transformations in 2D</li> <li>Defining central symmetry, orthogonal projection of a vector, identical transformation</li> <li>Defining a rotation through an angle about the origin,</li> <li>Showing that a linear transformation is isomorphism in 2D or not</li> <li>Performing operations on linear transformations in 2D</li> <li>Constructing the composite of two linear transformations in 2D</li> <li>Determining whether a linear transformation in 2D is isomorphism or not.</li> <li>Determining the analytic expression of the inverse of an isomorphism in 2D</li> </ul>	<ul style="list-style-type: none"> <li>Defining the order of a matrix</li> <li>Defining a linear transformation in 2D by a matrix</li> <li>Defining operations on matrices of order 2</li> <li>Showing that a square matrix of order 2 is invertible or not</li> <li>Reorganising data into matrices</li> <li>Determining the matrix of a linear transformation in 2D</li> <li>Performing operations on matrices of order 2</li> <li>Constructing the matrix of the composite of two linear transformations in 2D</li> <li>Constructing the matrix of the inverse of an isomorphism of <math>\mathbb{R}^2</math></li> <li>Determining the inverse of a matrix of order 2</li> </ul>

<p><b>Competences to be practised</b></p>	<ul style="list-style-type: none"> <li>• Problem-solving</li> <li>• Communication</li> <li>• Cooperation</li> <li>• Research</li> </ul>	<ul style="list-style-type: none"> <li>• Research</li> <li>• Cooperation</li> <li>• Communication</li> <li>• Problem-solving</li> </ul>	<ul style="list-style-type: none"> <li>• Critical thinking</li> <li>• Cooperation</li> <li>• Communication</li> <li>• Research</li> </ul>
<p><b>Attitudes and values</b></p>	<ul style="list-style-type: none"> <li>• Apply and transfer the skills of dot product, magnitude to other areas of knowledge</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate the importance and the use of operations on transformation in 2D</li> <li>• Show curiosity for the study of operations on transformations in 2D</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate the importance and the use of matrices in organising data</li> <li>• Show curiosity for the study of matrices of order 2</li> </ul>

	Unit 13: Points, straight lines and circles in 2D	Unit 14: Measures of dispersions	Unit 15: Combinatorics
<b>Number of periods</b>	21+ homework	7+ homework	18 + homework
<b>Classroom Organisation</b>	Whole class orientation; then group work.	Whole class orientation; individual work and then working in groups.	Whole class orientation; then working in groups.
<b>Equipment required</b>	Manila paper, graph paper, geometric instruments (ruler, T-square), digital technology instruments including calculators.	Manila papers, graph papers, ruler, digital technology including calculators	Manila papers, graph papers, ruler, digital components including calculators...
<b>Activities</b>	<ul style="list-style-type: none"> <li>• Defining the coordinate of a point in 2D</li> <li>• Defining a straight line knowing its:               <ul style="list-style-type: none"> <li>- 2 points</li> <li>- direction vector</li> </ul> </li> <li>• Representing a point and or a vector in 2D</li> <li>• Calculating the distance between two points in 2D and the mid-point of a segment in 2D</li> <li>• Determining equations of a straight line (Vector equation, parametric equation, Cartesian equation)</li> <li>• Applying knowledge to find the centre, radius, and diameter a circle and find out the equation.</li> <li>• Performing operations to determine the intersection of a circle and a line</li> </ul>	<ul style="list-style-type: none"> <li>• Defining the variance, standard deviation and coefficient of variation</li> <li>• Analysing and interpreting critically data and inferring conclusions.</li> <li>• Determining the measures of dispersion of a given statistical series.</li> <li>• Applying and explaining the standard deviation as the more convenient measure of the variability in the interpretation of data</li> <li>• Expressing the coefficient of variation as a measure of the spread of a set of data as a proportion of its mean.</li> </ul>	<ul style="list-style-type: none"> <li>• Defining the combinatorial analysis</li> <li>• Recognising whether repetition is allowed or not. and if order matters or not in performing a given experiment</li> <li>• Constructing Pascal's triangle</li> <li>• Distinguishing between permutations and combinations</li> <li>• Determining the number of permutations and combinations of "n" items, "r" taken at a time.</li> <li>• Using counting techniques to solve related problems.</li> <li>• Using properties of combinations</li> </ul>

<p><b>Competences to be practised</b></p>	<ul style="list-style-type: none"> <li>• Critical thinking</li> <li>• Cooperation</li> <li>• Communication</li> <li>• Research</li> </ul>	<ul style="list-style-type: none"> <li>• Critical thinking</li> <li>• Problem-solving</li> <li>• Cooperation</li> <li>• Communication</li> </ul>	<ul style="list-style-type: none"> <li>• Cooperation</li> <li>• Communication</li> <li>• Problem-solving</li> <li>• Research</li> </ul>
<p><b>Attitudes and values</b></p>	<ul style="list-style-type: none"> <li>• Appreciate that a point is a fixed position in a plane</li> <li>• Show concern patiently and with mutual respect in representations and calculations</li> <li>• Be accurate in representations and calculations</li> <li>• Manifest a team spirit and think critically in problem solving related to the position of straight lines in 2D</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate the importance of measures of dispersion in the interpretation of data</li> <li>• Show concern on how to use the standard deviation as measure of variability of data.</li> </ul>	<ul style="list-style-type: none"> <li>• Appreciate the importance of counting techniques</li> <li>• Show concern on how to use the counting techniques</li> </ul>

	<b>Unit 16: Elementary probability</b>
<b>Number of periods</b>	7+ homework
<b>Classroom organisation</b>	Whole class orientation; then group work.
<b>Equipment required</b>	Manila paper, graph paper, ruler, digital instruments including calculators
<b>activities</b>	<ul style="list-style-type: none"> <li>• Defining probability and explain probability as a measure of chance</li> <li>• Distinguishing between mutually exclusive and non-exclusive events and computing their probabilities</li> <li>• Using and applying properties of probability to calculate the number possible outcomes of occurring events under equally likely assumptions</li> <li>• Determining and explaining expectations from an experiment with possible outcomes</li> </ul>
<b>Competences to be practised</b>	<ul style="list-style-type: none"> <li>• Problem solving</li> <li>• Cooperation</li> <li>• Communication</li> <li>• Research</li> </ul>
<b>Attitudes and values</b>	<ul style="list-style-type: none"> <li>• Appreciate the use of probability as a measure of chance</li> <li>• Show concern on patience, mutual respect, tolerance and curiosity in the determination of the number of possible outcomes of a random experiment</li> </ul>



# Topic area: Trigonometry

## Sub-topic area: Trigonometric circle and identities

UNIT

# 1

## Fundamentals of trigonometry

**Number of periods: 26**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define sine, cosine, tangent, cosecant, secant and cotangent of any angle and know the special values</li><li>• Convert radians to degrees and vice versa</li><li>• Differentiate between complementary angles, supplementary angles and co-terminal angles.</li></ul>	<ul style="list-style-type: none"><li>• Represent graphically sine, cosine and tangent functions and, together with the unit circle, use them to relate values of any angle to the value of a positive acute angle.</li><li>• Use trigonometry, including the sine and cosine rules, to solve problems involving triangles.</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the relationship between the trigonometric values for different angles</li><li>• Verify reasonableness of answers to exercises when solving problems</li></ul>

### Content

1. Trigonometric concepts
  - Angle and its measurements
  - Unit circle
  - Trigonometric ratios
  - Trigonometric identities

2. Reduction to functions of positive acute angles
3. Triangles and applications:
  - Bearing
  - Air navigation
  - Inclined plane

## Materials required

Geometrical instruments: rule, T-square, compass, protractors; graph papers, digital instruments such as calculators.

## Generic competences

- Communication
- Problem-solving
- Research
- Cooperation
- Critical thinking

## Cross-cutting issues

- **Gender studies**  
Groups to be composed of mixed gender.
- **Peace and values education**  
Learners work together harmoniously.
- **Inclusive education**  
Learners of different abilities work together and assist each other.

## Sample lesson plan

School: .....

Teacher's name: ...

.....

Term	Date	Subject	Class	Unit N°	Lesson N°	Duration	Class size
1	8 February 2016	Mathematics	S4	1	4	40	45
Type of special educational needs and number of learners							
Topic area		Trigonometry					
Sub-topic area		Trigonometric circles and identities					
Unit title		Fundamentals of trigonometry					
Key unit competence		Use trigonometric circles and identities to determine trigonometric ratios and apply them to solve related problems					
Title of the lesson		Trigonometric ratios					
Instructional objective		Using geometric instruments, graph paper and pencils the learners will be able to derive the trigonometric ratios of the special angles $30^\circ$ and $60^\circ$ in 20 minutes, with accuracy.					
Plan for this class (location: in / outside)		In the classroom					
Learning materials (for all learners)		Geometric instruments (ruler, T-square, protractor ), graph paper, pencils, flash cards					
References		Achievers Mathematics Senior 4 Student's Book Mathematics syllabus by REB.					

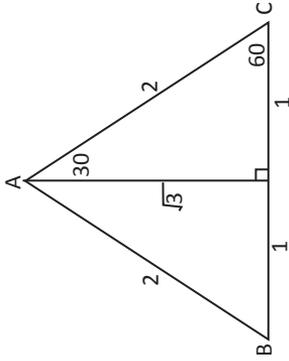
<p>Timing for each step</p>	<p>Description of teaching and learning activity</p> <p>Teacher guides learners in recalling properties of equilateral triangles and construction to discern trigonometric ratios of the angles <math>30^\circ</math> and <math>60^\circ</math>.</p> <p>Learners draw an equilateral triangle, ABC, of sides 2 units in length; the draw a line AD from A perpendicular to BC. AD bisects BC giving <math>BD=CD=1</math>. They complete table of the trigonometric ratios (cos, sin and tan) of angles <math>30^\circ</math> and <math>60^\circ</math>.</p>		<p><b>Generic competences and cross-cutting issues to be addressed</b></p>
<p><b>Introduction</b></p> <p>8 min</p>	<p><b>Teacher activities</b></p> <ul style="list-style-type: none"> <li>- Give students samples of equilateral triangles on flash cards</li> <li>- Ask learners the properties of equilateral triangles – size and angles.</li> <li>- Ensure the learner with visual impairment sits in front of the class.</li> </ul>	<p><b>Learner activities</b></p> <ul style="list-style-type: none"> <li>- Learners will talk about the angle sizes and sides of equilateral triangles.</li> <li>- They will measure and discern from flash cards that the angles and sides are equal.</li> </ul>	<p><b>Generic competences:</b></p> <ul style="list-style-type: none"> <li>- <b>Communication</b> - learners discuss angle sizes and sides of equilateral triangles.</li> <li>- <b>Problem - solving</b> - learners measure angles and sides accurately</li> </ul> <p><b>Cross-cutting issues:</b></p> <ul style="list-style-type: none"> <li>- <b>Peace and values education:</b> Learners take turns when answering questions and respect the opinions of others.</li> </ul>

**Development of the lesson**

27 min

1. Ensure learners are in pairs and have the required materials for construction.
2. Ask learners to follow the procedure of Activity 1.8 of the Student's Book.
3. Move round the class guiding them.
4. Let groups present their findings.
5. Write the main points from what learners have presented.

1. In pairs, learners will draw an equilateral triangle,  $ABC$ , of sides 2 units in length. Next they draw a line  $AD$  from  $A$  perpendicular to  $BC$ .  $AD$  bisects  $BC$  giving  $BD = DC = 1$ .



2. They use Pythagoras' theorem to determine the height  $AD = \sqrt{3}$ .

**Generic competences:**

- Communication** - learners present their findings to the rest of the class.
- Cooperation** - learners work together harmoniously on assigned tasks.
- Problem - solving** - learners draw triangles and lines accurately. They then determine the ratios of the sides.

		<p>3. They then determine the ratios of sides taking into account the angles.</p> <p>4. Learners then present the findings in class, and fill in a table like the following.</p>	<p><b>Cross-cutting issues</b></p> <ul style="list-style-type: none"> <li>• <b>Standardization culture:</b> Learners carry out accurate constructions and measurements, and adhere to the rules of trigonometric ratios.</li> </ul>						
	<p>6. Ensure the learner with visual impairment is paired with one who has good vision.</p>	<p><b>Trigonometric ratios for the special angles 30° and 60°</b></p> <table border="1" data-bbox="571 609 799 1075"> <tr> <td><math>\sin 60^\circ = \frac{\sqrt{3}}{2}</math></td> <td><math>\sin 30^\circ = \frac{1}{2}</math></td> </tr> <tr> <td><math>\cos 60^\circ = \frac{1}{2}</math></td> <td><math>\cos 30^\circ = \frac{\sqrt{3}}{2}</math></td> </tr> <tr> <td><math>\tan 60^\circ = \sqrt{3}</math></td> <td><math>\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}</math></td> </tr> </table>	$\sin 60^\circ = \frac{\sqrt{3}}{2}$	$\sin 30^\circ = \frac{1}{2}$	$\cos 60^\circ = \frac{1}{2}$	$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\tan 60^\circ = \sqrt{3}$	$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	<ul style="list-style-type: none"> <li>• <b>Peace and values education:</b> Learners take turns when presenting their findings and respect the opinions of others.</li> </ul>
$\sin 60^\circ = \frac{\sqrt{3}}{2}$	$\sin 30^\circ = \frac{1}{2}$								
$\cos 60^\circ = \frac{1}{2}$	$\cos 30^\circ = \frac{\sqrt{3}}{2}$								
$\tan 60^\circ = \sqrt{3}$	$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$								

<p><b>Conclusion</b> 5 min</p>	<ol style="list-style-type: none"> <li>The teacher gives an oral summary of the ratios of the special angles <math>30^\circ</math> and <math>60^\circ</math>.</li> <li>Teacher asks learners, at random, to name the different ratios <ul style="list-style-type: none"> <li>– cosine, sine and tangent of <math>30^\circ</math> and <math>60^\circ</math>.</li> </ul> </li> <li>Gives learners homework – to write the complete table of trigonometric ratios of angles <math>0^\circ, 30^\circ, 60^\circ, 45^\circ, 60^\circ</math> and <math>90^\circ</math>.</li> </ol>	<ol style="list-style-type: none"> <li>The learners answer the oral questions posed by the teacher; they do this in an orderly manner by raising their hands and waiting to be picked by the teacher.</li> <li>They each draw the summary table showing the trigonometric ratios of the angles <math>30^\circ</math> and <math>60^\circ</math>.</li> </ol>	<p><b>Generic competences:</b></p> <ul style="list-style-type: none"> <li>- <b>Communication</b> - learners answer oral questions.</li> <li>- <b>Problem-solving</b> - learners come up with the summary table for trigonometric ratios.</li> </ul> <p><b>Cross-cutting issue:</b></p> <ul style="list-style-type: none"> <li>- <b>Peace and values education</b> - learners take turns when answering questions and respect the opinions of others.</li> </ul>
<p>Teacher self-evaluation</p>			

## Teaching and learning activities

### Introductory activity

Ask each student to imagine a point on the edge of a wheel. As the wheel turns how high is the point above the centre? Guide them to sketch the graph.

### Main activities

1. Introduce the topic by giving some examples of angles and their measurements.  
Guide learners in carrying out Activity 1.1 on page 1 of the Student's Book. Ensure the groups are working together harmoniously and that all learners are included irrespective of gender and ability. Assist them in coming up with the explanation of what a radian is. Encourage them to appreciate each others' effort.
2. Let them attempt Activity 1.2 on page 2 to introduce the concept of radians. Guide learners in converting angles from degrees to radians and vice versa. Use adequate examples. Then let them attempt Task 1.1 on page 4 of the Student's Book.
3. Ask students to research on the trigonometric ratios. They will start with the use of the unit circles. Guide them in Activities 1.3 and 1.4 on page 5 of the Student's Book. Let them discuss and give the definitions for the ratios – sine, cosine, tangent, cosecant, secant and cotangent.
4. Use triangles to assist learners define trigonometric ratios related to  $30^\circ$ ,  $45^\circ$  and  $60^\circ$ . Let them verify them using Activities 1.5, 1.6, 1.7 and 1.8 on pages 7, 8 and 9 of the Student's Book.
5. Take them through the different types of angles on pages 9 to 13. Let them attempt Tasks 1.2, 1.3 and 1.4 on pages 13 and 14 of the Student's Book.
6. Explain the addition formula. Let them do Tasks 1.5, 1.6 and 1.7 on pages 16 and 17 of the Student's Book.
7. Let them work in groups to draw the trigonometric graphs of Activities 1.9, 1.10 and 1.11 of pages 18 and 19 of the Student's Book. Discuss with students how to read and plot the functions of sine and of cosine. Guide them in plotting graphs.
8. Help the students to use trigonometry, including the sine and cosine rules, to solve problems involving triangles. Ask them to state where we can apply trigonometric ratios. This will make them appreciate the relationship between trigonometric values of different angles. Let them attempt Tasks 1.8, 1.9 and 1.10 on pages 22, 24 and 28 respectively, of the Student's Book.

## Extra practice for learners

1. Use geometrical instruments to measure:
  - The size of a right angle
  - The size of a straight angle
  - The size of a full turn

### Solution

- The measure of a right angle is  $\frac{1}{4}$  turn which is  $90^\circ$ ;
- The measure of a straight angle is  $\frac{1}{2}$  turn which is  $180^\circ$ ;
- The measure of a full turn which is  $360^\circ$ .

2. Convert  $\frac{7\pi}{6}$  radians to degrees and  $60^\circ$  to radians.

### Solution

$$\frac{7}{6} \text{ radians} = \frac{7\pi}{6} \left( \frac{180}{\pi} \right) = 120$$

$$60^\circ = 60^\circ \left( \frac{\pi}{180} \right) = \frac{\pi}{3} \text{ radians}$$

## Reinforcement activity

On graph paper draw circle radius 1 cm and measure half chord length and distance from centre to chord for angles (say multiples of  $15^\circ$ ) -plot the graphs- use calculator to determine which is sine and cosine. What is the radius of the calculator's circle?-unit circle.

## Learners of varying strengths and abilities

### a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

### b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for

participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

## Additional tasks

### Task IA - For slow learners

- Simplify the following
  - $\cos^2 x \tan^2 x + \sin^2 x \cot^2 x$
  - $\sin^2 x + \sin^2 x \cot^2 x$
- Prove the following identity

$$\frac{\sin x}{1 + \cos x} + \frac{\sin x}{1 - \cos x} = 2 \csc x$$

### Answers

$$1. \quad (a) \quad \cos^2 x \tan^2 x + \sin^2 x \cot^2 x = \cos^2 x \frac{\sin^2 x}{\cos^2 x} + \sin^2 x \frac{\cos^2 x}{\sin^2 x} = \sin^2 x + \cos^2 x = 1$$

$$(b) \quad \sin^2 x + \sin^2 x \cot^2 x = \sin^2 x (1 + \cot^2 x) = \sin^2 x \left( 1 + \frac{\cos^2 x}{\sin^2 x} \right) \\ = \sin^2 x \left( \frac{\sin^2 x + \cos^2 x}{\sin^2 x} \right) = \frac{\sin^2 x}{\sin^2 x} = 1$$

$$2. \quad \frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{\sin^2 x + (1 + \cos x)^2}{\sin x(1 + \cos x)} = \frac{\sin^2 x + 1 + 2 \cos x + \cos^2 x}{\sin^2 x} = \\ \frac{2 + 2 \cos x}{\sin x(1 + \cos x)} \\ = \frac{2(1 + \cos x)}{\sin x(1 + \cos x)} = \frac{2}{\sin x} = 2 \left( \frac{1}{\sin x} \right) = 2 \csc x$$

### Task IB - For talented learners

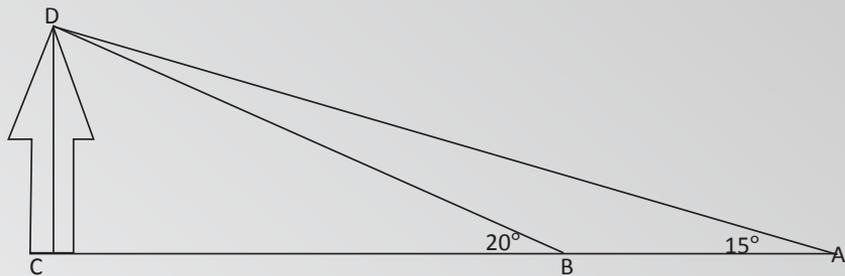
- From a tower 92 m high, two rocks which are in a horizontal line through the base of the tower are observed at angles of depression of  $15^\circ$  and  $20^\circ$ . Find the distance between the rocks if they are on:
  - the same side of the tower
  - opposite sides of the tower.

2. Two vertical lamp posts of equal height stand on either side of a roadway which is 30 m wide. At a point in the roadway between the lamp posts, the angles of elevation for which the tops of the lamp posts are observed are  $48^\circ$  and  $42^\circ$ . Determine
- the height of each lamp post.
  - the position of the point of observation.

### Answers

1.

a)



From the figure above,  $CD = 92\text{m}$

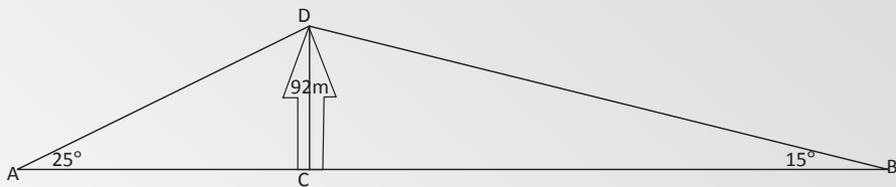
$$\text{In the triangle ACD, } \tan 15^\circ = \frac{92}{AC} \Rightarrow AC = \frac{92}{\tan 15^\circ} = 343.35\text{m}$$

$$\text{In the triangle BCD, } \tan 20^\circ = \frac{92}{CB} \Rightarrow CB = \frac{92}{\tan 20^\circ} = 252.77\text{m}$$

$$AB = AC - BC = 343.35\text{ m} - 252.77\text{ m} = 90.58\text{ m}.$$

If the rocks are on the same side of the tower, the distance between them is 90.58 m.

b)



From the figure above,  $CD = 92\text{ m}$

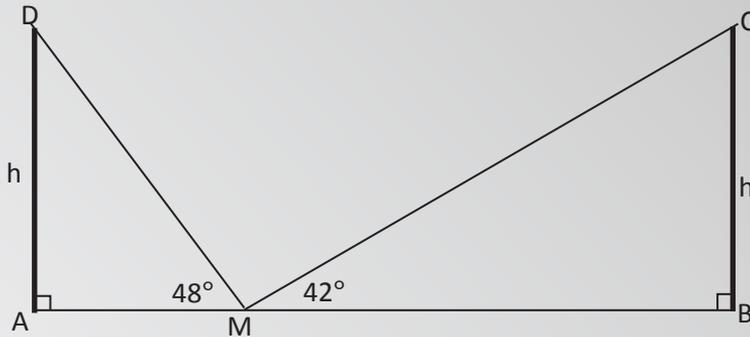
$$\text{In the triangle BCD, } \tan 15^\circ = \frac{92}{CB} \Rightarrow CB = \frac{92}{\tan 15^\circ} = 343.35\text{ m}$$

$$\text{In the triangle ACD, } \tan 20^\circ = \frac{92}{AC} \Rightarrow AC = \frac{92}{\tan 20^\circ} = 252.77\text{ m}$$

$$AB = AC + BC = 252.77\text{ m} + 343.35\text{ m} = 596.12\text{ m}$$

If the rocks are on the opposite sides of the tower, the distance between them is 596.12 m.

2. **Illustration**



From the figure above, the lamp posts are at A and B. The point of observation at M.

We have  $AB = 30$ . Let  $AM = x$  then  $MB = 30 - x$

$$\tan 48^\circ = \frac{h}{AM} = \frac{92}{\tan 20^\circ} \Rightarrow h = x \tan 48^\circ \Rightarrow h = 1.11x$$

$$\tan 42^\circ = \frac{h}{MB} = \frac{h}{30 - x} \Rightarrow h = (30 - x) \tan 42^\circ \Rightarrow h = 27.01 - 0.90x$$

Let us solve for x

$$1.11x = 27.01 - 0.90x$$

$$2.01x = 27.01$$

$$x = 13.44$$

a) The point is on 13.44 m from the lamp post at A

b)  $h = 1.11 \times 13.44 = 14.92$

The height of each lamp post is 14.92 m

## Additional information

### Proof of trigonometric ratios formulas

1. **Prove that**

$$1 + \tan^2 \theta = \sec^2 \theta \quad (\cos \theta \neq 0)$$

**Proof:**

$$\text{From } \sin^2 \theta + \cos^2 \theta = 1.$$

Dividing both sides by  $\cos^2 \theta$  gives

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Which simplifies to  $1 + \tan^2 \theta = \sec^2\theta$  ( $\cos \theta \neq 0$ )

2. **Prove that**

$$1 + \cot^2 \theta = \csc^2 \theta \quad (\sin \theta \neq 0)$$

**Proof**

From  $\sin^2 \theta + \cos^2 \theta = 1$ .

Dividing both sides by  $\sin^2$  gives

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

Which simplifies to  $1 + \cot^2 \theta = \csc^2 \theta$  ( $\sin \theta \neq 0$ )

3. **Prove that:**

$\tan \theta + \cot \theta = \sec \theta \csc \theta$  whenever both sides have meaning

**Proof:**

LHS =  $\tan \theta + \cot \theta$

$$= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta}$$

$$= \frac{1}{\cos \theta \sin \theta} = \frac{1}{\cos \theta} \frac{1}{\sin \theta}$$

=  $\sec \theta \csc \theta =$  RHS

Provided  $\sin \theta \neq 0$ , i.e.  $\theta \neq \frac{k\pi}{2}$ ,  $k \in \mathbb{Z}$ .

4. **Prove that:**

$$\sin^2 A - 4 \cos^2 A + 1 = 2 \sin^2 A - 3 \cos^2 A = 3 \sin^2 A - 2 \cos^2 A - 1$$

**Proof:**

$$\begin{aligned} \text{LSH} - \sin^2 A - 4 \cos^2 A + (\sin^2 A + \cos^2 A) &= 2 \sin^2 A - 3 \cos^2 A = \text{MIDDLE} = \\ 2 \sin^2 A - 2 \cos^2 A - \cos^2 A &= 2 \sin^2 A - 2 \cos^2 A - (1 - \sin^2 A) = 3 \sin^2 A - 2 \\ \cos^2 A - 1 &= \text{RHS}. \end{aligned}$$

5. **Prove that:**

$$\frac{\cos^2 A}{1 + \tan^2 A} - \frac{\sin^2 A}{1 + \tan^2 A} = 1 - 2 \sin^2 A \text{ whenever both sides have meaning.}$$

**Proof:**

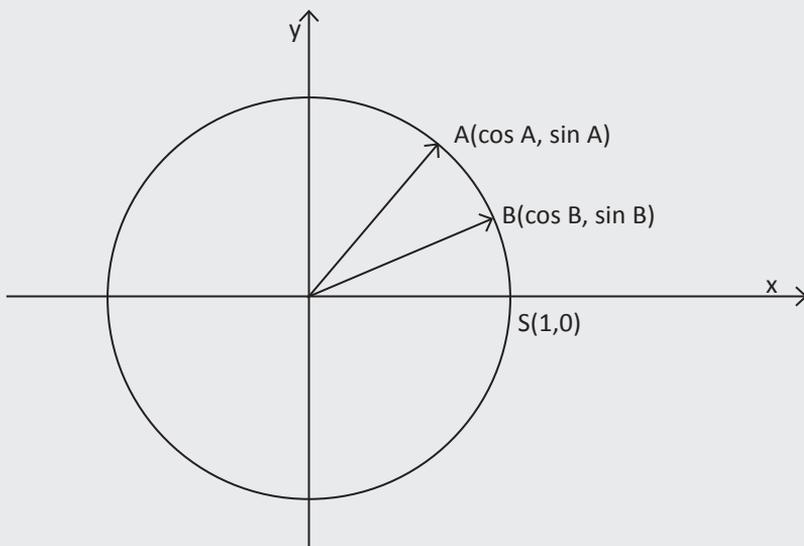
$$\text{LSH} = \frac{\cos^2 A}{\sec^2 A} - \frac{\sin^2 A}{\csc^2 A} = \cos^4 A = (\cos^2 A - \sin^2 A)(\cos^2 A + \sin^2 A) = \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A = \text{RHS}$$
 Provided  $\tan A$  and  $\cot A$  both exist, i.e.  $\cos A \neq 0$  and  $\sin A \neq 0$ . Thus both sides have meaning provided  $A \neq \frac{k\pi}{2}, k \in \mathbb{Z}$ .

**Additional formulae**

1.  $\cos(A - B) = \cos A \cos B + \sin A \sin B$
2.  $\cos(A + B) = \cos A \cos B - \sin A \sin B$
3.  $\sin(A + B) = \sin A \cos B + \cos A \sin B$
4.  $\sin(A - B) = \sin A \cos B - \cos A \sin B$

**Proof of addition formulae**

1. Consider the points  $A(\cos A, \sin A)$  and  $B(\cos B, \sin B)$  on the unit circle.



$\text{AOS} = A$  and  $\text{BOS} = B$  and  $\text{AOB} = A - B$ .

$$\cos(A - B) = \frac{\vec{OA} \cdot \vec{OB}}{|\vec{OA}| |\vec{OB}|} = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \cdot \begin{pmatrix} \cos B \\ \sin B \end{pmatrix} \text{ since } |\vec{OA}| = |\vec{OB}| = 1$$

Thus  $\cos(A - B) = \cos A \cos B + \sin A \sin B$ .

2. Replace  $B$  with  $-B$  in 1,

$$\cos(a + B) = \cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

3. Replace A with  $\frac{\pi}{2} + A$  in 2.

$$\cos\left(\frac{\pi}{2} + A + B\right) = \cos\left(\frac{\pi}{2} + A\right) \cos B - \sin\left(\frac{\pi}{2} + A\right) \sin A$$

$$-\sin(A + B) = \sin A \cos B - \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

4. Replace A with  $-B$  in 3.

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

### Additional formulae for tangent function

$$1. \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$2. \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

### Proof of Addition formulae for the tangent function

$$1. \quad \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$2. \quad \text{Put } -B \text{ for } B \text{ in 1 } \tan(A - B) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

## Assessment criteria

Apply trigonometric concepts to solve problems involving triangles and angles

## Answers to Tasks of Unit 1 in the Student's Book

### Task 1.1 on page 4

- |                      |                      |                      |                        |                      |
|----------------------|----------------------|----------------------|------------------------|----------------------|
| (a) $\frac{\pi}{2}$  | (b) $\frac{\pi}{3}$  | (c) $\frac{\pi}{6}$  | (d) $\frac{\pi}{10}$   | (e) $\frac{\pi}{20}$ |
| (f) $\frac{3\pi}{4}$ | (g) $\frac{5\pi}{4}$ | (h) $\frac{3\pi}{2}$ | (i) $2\pi$             | (j) $4\pi$           |
| (l) $3\pi$           | (m) $\frac{\pi}{5}$  | (n) $\frac{4\pi}{9}$ | (o) $\frac{23\pi}{18}$ |                      |
- |                 |                 |                 |                 |                  |
|-----------------|-----------------|-----------------|-----------------|------------------|
| (a) $36^\circ$  | (b) $108^\circ$ | (c) $135^\circ$ | (d) $10^\circ$  | (e) $20^\circ$   |
| (f) $140^\circ$ | (g) $18^\circ$  | (h) $27^\circ$  | (i) $150^\circ$ | (j) $22.5^\circ$ |

3. (a)

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$

(b)

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$

### Task 1.2 on page 13

1.

	$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$
(a)	$0^\circ$	0	1	0
(b)	$90^\circ$	1	0	–
(c)	$45^\circ$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
(d)	$120^\circ$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$

2. (a)  $\cos \theta = \pm 0.6$ ,  $\tan \theta = \pm \frac{4}{3}$ , (c)  $\sin \theta = \pm \frac{2}{\sqrt{5}}$ ,  $\cos \theta = \mp \frac{1}{\sqrt{5}}$

(b)  $\sin \theta = \pm \frac{\sqrt{3}}{2}$ ,  $\tan \theta = \pm \sqrt{3}$

3. (a)  $\sin \theta$  (b)  $\cos \theta$  (c)  $2 \cos \theta$

(d)  $2 \sin \theta$  (e)  $2 \cos \theta$

4. (a) Any 4 multiples of  $180^\circ$ .

(b) Any 4 odd multiples of  $90^\circ$ .

(c) Any 4 multiples of  $90^\circ$ .

### Task 1.3 on page 14

1. Prove that

$$1 + \tan^2 \theta = \sec^2 \theta \quad (\cos \theta \neq 0)$$

Proof

From  $\sin^2 \theta + \cos^2 \theta = 1$ .

Dividing both sides by  $\cos^2 \theta$  gives

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

Which simplifies to  $1 + \tan^2 \theta = \sec^2 \theta$  ( $\cos \theta \neq 0$ )

2. Prove that

$$1 + \cot^2 \theta = \csc^2 \theta \quad (\sin \theta \neq 0)$$

Proof

From  $\sin^2 \theta + \cos^2 \theta = 1$ .

Dividing both sides by  $\sin^2 \theta$  gives

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

Which simplifies to  $1 + \cot^2 \theta = \csc^2 \theta$  ( $\sin \theta \neq 0$ )

3. Prove that:

$\tan \theta + \cot \theta = \sec \theta \csc \theta$  whenever both sides have meaning

Proof:

LHS =  $\tan \theta + \cot \theta$

$$\begin{aligned} &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{1}{\cos \theta \sin \theta} + \frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta} \end{aligned}$$

Provided  $\sin \theta \neq 0$  and  $\cos \theta \neq 0$ , i.e.,  $\theta \neq \frac{k\pi}{2}$ ,  $k \in \mathbb{Z}$ .

4. Prove that:

$$\sin^2 A - 4 \cos^2 A + 1 = \sin^2 A - 3 \cos^2 A = 3 \sin^2 A - 2 \cos^2 A - 1$$

Proof

$$\begin{aligned} \text{LHS} &= \sin^2 A - 4 \cos^2 A + (\sin^2 A + \cos^2 A) = 2 \sin^2 A - 3 \cos^2 A = \text{MIDDLE} = \\ &= 2 \sin^2 A - 2 \cos^2 A - \cos^2 A = 2 \sin^2 A - 2 \cos^2 A - (1 - \sin^2 A) = \\ &= 3 \sin^2 A - 2 \cos^2 A - 1 = \text{RHS}. \end{aligned}$$

5. Prove that :

$$\frac{\cos^2 A}{1 + \tan^2 A} + \frac{\sin^2 A}{1 + \tan^2 A} = 1 - 2 \sin^2 A \text{ whenever both sides have meaning.}$$

Proof

$$\begin{aligned} \text{LHS} &= \frac{\cos^2 A}{\sec^2 A} - \frac{\sin^2 A}{\sec^2 A} = \cos^4 A - \sin^4 A = (\cos^2 A - \sin^2 A) (\cos^2 A + \sin^2 A) \\ &= \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A = \text{RHS} \text{ Provided } \tan \end{aligned}$$

A and  $\cot A$  both exist, i.e.  $\cos A \neq 0$  and  $\sin A \neq 0$ . Thus both sides have meaning provided,  $A \neq \frac{k\pi}{2}$ ,  $k \in \mathbb{Z}$ .

**Task 1.4 on page 14**

- |                |                          |                     |                     |
|----------------|--------------------------|---------------------|---------------------|
| (a) 1          | (b) $\sec^2 \frac{A}{4}$ | (c) 1               | (d) $\csc^2 \theta$ |
| (e) 1          | (f) 1                    | (g) 1               | (h) 2               |
| (i) $\cos^2 A$ | (j) $\sin^2 2B$          | (k) $\tan^2 \theta$ | (l) $-\cot^2 A$     |

**Task 1.5 on page 16**

- $\sin 2x \cos y + \cos 2x \sin y$
  - $\sin 3B \cos 40^\circ + \cos 3B \sin 40^\circ$
  - $\sin 2A \cos 2B + \cos 2A \sin 2B$
  - $\sin x \cos 2y - \cos x \sin 2y$
  - $\frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta$
  - $\cos x \cos \frac{1}{2}y - \sin x \sin \frac{1}{2}y$
  - $\cos 3A \cos 3B - \sin 3A \sin 3B$
  - $\frac{\sqrt{3}}{2} \cos A + \frac{1}{2} \sin A$
  - $\frac{1}{2} \cos A - \frac{\sqrt{3}}{2} \sin A$
  - $\cos B \cos C + \sin B \sin C$
  - $\sin B$
  - $\frac{1}{2} \cos 2A - \frac{\sqrt{3}}{2} \sin 2A$
- $-\frac{24}{25}$
  - 1
  - 0
- $\frac{\sqrt{2} - \sqrt{6}}{4}$
  - $\frac{\sqrt{6} - \sqrt{2}}{4}$
  - $\frac{\sqrt{6} - \sqrt{2}}{4}$
  - $\frac{\sqrt{6} - \sqrt{2}}{4}$

**Task 1.6 on page 17**

- 1
  - $\cot(A - B)$
  - $\frac{\sqrt{3}}{3}$
  - $\tan(\frac{\pi}{4} + A)$
  - $\tan(\frac{\pi}{4} + A)$
  - $\tan 2A$

**Task 1.7 on page 17**

- 0
- $\frac{1}{2}$
- $\frac{1}{4}(\sqrt{6} - \sqrt{2})$

4.  $-(2 + \sqrt{3})$                       5.  $\frac{1}{4}(\sqrt{6} - \sqrt{2})$                       6.  $\frac{1}{4}(\sqrt{6} - \sqrt{2})$   
 7.  $\sin 3\theta$                               8. 0                                      9.  $\tan 3A$   
 10.  $\tan \beta$   
 11. (a)  $\frac{3}{5}$                               (b)  $\frac{-4}{5}$                               (c)  $\frac{-3}{5}$

**Task 1.8 on page 22**

1. (a)  $x = 4.44$  cm                      (b)  $x = 21.49$  cm                      (c)  $x = 4$  cm  
 2. (a)  $A = 90^\circ$ ,  $B = 53.13^\circ$ ,  $C = 36.86^\circ$   
 (b)  $C = 53.13^\circ$ ,  $B = 90^\circ$   
 3. (a)  $A = 55.77^\circ$ ,  $B = 41.4^\circ$ ,  $C = 82.81^\circ$   
 (b)  $A = 57.12^\circ$ ,  $B = 44.41^\circ$ ,  $C = 78.64^\circ$   
 4. (a)  $c = 4.31$  cm,  $A = 55.11^\circ$ ,  $B = 79.87^\circ$   
 (b)  $b = 13.11$  cm,  $A = 49.96^\circ$ ,  $C = 73.18^\circ$   
 (c)  $b = 5.21$  cm,  $A = 35.5^\circ$ ,  $C = 27.5^\circ$   
 (d)  $b = 13.7$  cm,  $A = 49.1^\circ$ ,  $C = 70.9^\circ$

**Task 1.9 on page 24**

1. (a)  $x = 5.45$  cm                      (b)  $x = 5.72$  m  
 (c)  $x = 3.18$  cm  
 2. (a)  $b = 17.59$  cm,  $A = 58^\circ$                       (b)  $a = 11.62$  cm  $c = 11.62$  cm  
 3. (a)  $b = 7.07$  cm                      (b)  $c = 6.06$  cm                      (c)  $B = 15.66^\circ$   
 4.  $C = 62.1^\circ$  or  $C = 117.9^\circ$   
 5. (a)  $A = 49.5^\circ$                       (b)  $C = 44.3^\circ$   
 (c)  $B = 72.05^\circ$  or  $107.95^\circ$

**Task 1.10 on page 28**

1. The tree was 8.91 m tall.  
 2. The distance is 369.15 m  
 3. The two cars are 60.13 km apart.  
 4. The distance is 45.7 m  
 5. The bearing of B from A is  $S54^\circ 10' W$  and the distance of B from A is 22.2 km.  
 6. The height of the hill is 763.94 m.

7. The length of the field is 2460.36 m.
8. Height of the tree is 34.64 m and the breadth of the river is 20 m.
9. The height of the cliff is 107.96 m.
10. The distance between the bill-boards is
  - (a) 6.14 m.
  - (b) 49.82 m
11. 342 m
12. 384 km per hour

# Topic area: Algebra

## Sub-topic area: Mathematical logic and applications

UNIT

# 2

## Propositional and predicate logic

**Number of periods: 14**

### Learning objectives

By the end of this topic, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Distinguish between statements and propositions</li><li>• Convert into logical formula composite propositions, and vice versa</li><li>• Draw the truth table of a composite proposition</li><li>• Recognise the most often used tautologies (example: De Morgan's Laws)</li></ul>	<ul style="list-style-type: none"><li>• Use mathematical logic to infer conclusions from given propositions</li><li>• Show that a given logic statement is tautology or a contradiction</li></ul>	<ul style="list-style-type: none"><li>• Judge situations accurately and act with equality</li><li>• Observe situations and make appropriate decisions</li><li>• Appreciate and act with thoughtfulness: grasp and demonstrate carefulness.</li><li>• Develop and show mutual respect</li><li>• Demonstrate broadmindedness</li></ul>

### Content

- 1 Introduction and fundamental definitions
2. Propositional logic:
  - Truth tables

- Logical connectives
  - Tautologies and contradictions
3. Predicate logic:
    - Propositional functions
    - Quantifiers
  4. Applications:
    - Set theory
    - Electric circuits.

## Materials required

Manila papers, markers, rulers

## Generic competences

- Communication
- Research
- Cooperation
- Problem solving
- Critical thinking

## Cross-cutting issues

- **Gender studies**

Learners work together in groups composed of both gender. The examples given in the Student's Book are inclusive of both gender.

## Teaching and learning activities

### Introductory activity

Group investigation: let the learners research in advance in the library about proposition logic.

### Reinforcement activity

Learners research on the difference between a statement and a proposition. They also research further on propositional logic and its application. Give them Activity 2.1 on page 31 of the Student's Book to work on.

## Main activities

1. Introduce the unit by giving some examples of statements and propositions. Guide students to give the definitions for statement and proposition. Let them attempt, in pairs, Task 2.1 on page 32 of the Student's Book.
2. Let them, in groups, convert logical formulae to composite propositions and vice versa. Guide them on how they can use mathematical logic to infer conclusions from given propositions.
3. Explain to them what truth tables are. Give them Activity 2.2 on page 34 and then guide them in drawing truth tables of composite propositions. Let them attempt Tasks 2.2 on page 36 of the Student's Book. Ensure they get these correct.
4. Let the students do Activity 2.3 as found on page 34 of the Student's Book. Discuss with students if a given logic statement is tautology or a contradiction. Guide them in recognition of the most commonly used tautologies, that is De Morgan's Laws. Let them work in pairs to do Tasks 2.3 and 2.4 on pages 39 and 41 respectively of the Student's Book. Let them do Activity 2.4 on page 42.
5. Introduce logical quantifiers. Let them attempt Activity 2.5 on page 42 of the Student's Book.
6. Introduce the students to applications of proposition and predicate logic. Let them attempt Activities 2.6 and 2.7 on page 43 of the Student's Book.
7. Tasks 2.5 and 2.6 on pages 44 and 46 respectively should be adequate to assist them practise on Applications.
8. Guide them to develop skills and attitudes such as accurate judgement and fairness.

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. Give them individual exercises to work on as homework.

### (b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity

of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

## Additional tasks

### Task 2A - For slow learners

- Use a truth table to show that the following statement is logically equivalent:

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

- Negate the statement  $\forall x \in \mathbb{R}: x^2 \geq 0$

### Answers

- $\sim(p \wedge q) \equiv \sim p \vee \sim q$

P	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

- $\exists x \in \mathbb{R}: x^2 < 0$

### Task 2B - For talented learners

- Find the negations of the following proposition
  - $\forall x \in D, P(x)$
  - $\exists x \in D, P(x)$
- Let  $P(x,y): x = y + 3$  with domain the collection of natural numbers,  $\mathbb{N}$ . Find the truth values of the propositions
  - For  $P(5, 2)$  and
  - $P(6, 4)$
  - $P(12, 9)$

## Answers

1. a)  $\sim[\forall x \in D, P(x)] = \exists x \in D, \sim P(x)$   
b)  $\sim[\exists x \in D, P(x)] = \forall x \in D, \sim P(x)$
2. By substitution in the expression of P, we find
  - a) For P(5, 2),  $5 = 2 + 3$ . Therefore P(5, 2) is true
  - b) For P(6, 4),  $6 \neq 4 + 3$ . Therefore P(5, 2) is false
  - c) For P(12, 9),  $12 = 9 + 3$ . Therefore P(5, 2) is true

## Additional information

### Additional example

Deduce if the given statement is or is not a proposition

- r:  $4 < 8$
- s: If  $x = 4$  then  $x + 3 = 7$
- t: Nyanza is a chief city of Rwanda
- p: What a beautiful evening!

### Solution

The statements r, s, t are logic propositions while statements p is not a logic proposition.

### Venn diagrams

Venn diagrams can also be used to check the validity of arguments. An argument is the assertion that statement S follows from other statements  $S_1, S_2, \dots$  etc. Statements  $S_1, S_2, \dots$  etc. are called premises or hypothesis and the statement S is called the conclusion.

Note that the argument consists of two parts. One part of an argument consists of all hypothesis and the other part contains the conclusion derived from the hypothesis.

### Proof of De Morgan's Law

1.  $\sim(p \vee q) = \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

2.  $\sim(p \wedge q) = p \vee \sim q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

## Assessment criteria

Learner is able to use mathematical logic:

- to organise scientific knowledge
- as a tool of reasoning and argumentation in daily life.

## Answers to Tasks of Unit 2 in the Student's Book

### Task 2.1 on page 32

- Proposition since it is a declarative sentence.
- Proposition since it is a declarative sentence.
- Proposition since it is a declarative sentence.
- Not proposition since it is not a declarative sentence.
- Proposition since it is a declarative sentence.
- Proposition since it is a declarative sentence.

### Task 2.2 on page 36

1. a)

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

b)

$p$	$\sim p$	$p \wedge (\sim p)$	$\sim [p \wedge (\sim p)]$
T	F	F	T
F	T	F	T

c)

$p$	$\sim q$	$\sim q$	$p \wedge \sim q$
T	T	F	T
T	F	T	F
F	T	F	T
F	F	T	T

d)

$p$	$q$	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

e)

$p$	$q$	$\sim q$	$p \wedge (\sim q)$	$\sim [p \wedge (\sim q)]$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

f)

$p$	$q$	$r$	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

g)

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \wedge (\sim p)$
T	T	T	F	F
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

h)

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$\sim[(\sim p) \vee (\sim q)]$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

i)

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim p \wedge \sim q$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

2. a)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q \vee \sim q$	$\sim p (\sim q \wedge \sim q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

We see that the columns of  $\sim(p \wedge q)$  and  $\sim p \vee \sim q$  are identical.

b)

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$\sim(\sim p \wedge \sim q)$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	F

We see that columns of  $p \vee q$  and  $\sim(\sim p \wedge \sim q)$  are identical.

3. (a) This is a conjunction statement. So we combine by using the word “and” symbol  $\wedge$ .

Now: Let  $p$  = Tuyishimire plays football

$q$  = Tuyishimire plays netball

The statement is written as  $p \wedge q$  and read  $p$  and  $q$ .

Truth table for  $p \wedge q$ .

$p \wedge q$  is true only when  $p$  is true and  $q$  is true and otherwise it is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- (b) Let  $p$  be “I work hard” and  $q$  be “I will pass the examination”.

Then the sentence is written  $p \Rightarrow q$ .

Truth table for  $p \Rightarrow q$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(c) Let p be “A number is even”

q be “it is divisible by 2”

The statement is written as  $p \Leftrightarrow q$  and read as “p if and only if q”,  
p if q.

Truth table for  $p \Leftrightarrow q$ .

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

4. (a) Let p be Kalisa plays football, negation of p is the opposite of  $\sim p$  denoted as  $\sim p$ .

The truth table

p	$\sim p$
T	F
F	T

- (b) i.  $p \wedge \sim q$  = Nsengimana speaks Kinyarwanda and not French  
 ii.  $\sim(\sim p)$  = Nsengimana speaks Kinyarwanda.  
 iii.  $\sim p \vee \sim q$  = Nsengimana does not speak Kinyarwanda or French  
 iv.  $\sim p \wedge \sim q$  = Nsengimana does not speak neither Kinyarwanda nor French.

- (c) i. Truth table for  $\sim(\sim p \wedge q)$

p	q	$\sim p$	$\sim p \wedge q$	$\sim(\sim p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

- ii. Let p be “Today is Monday”

Let q be “Nyanza Football club is playing” in symbolic form the given compound statement is written as  $p \wedge \sim q$ .

5. Let p: you go to the market.

q: you will need money.

r: you will be able to buy something.

In symbolic form we have,  $p \Rightarrow (q \vee \sim r)$

The truth table of  $P \Rightarrow ((q \vee (\sim r))$

p	q	r	$\sim r$	$q \vee (\sim r)$	$p \Rightarrow ((q \vee (\sim r))$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	T	T

**Task 2.3 on page 39**

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

**Task 2.4 on page 41**

1. Let p: Murerwa is not home.

q: Iyakaremye is doing communal work

$p \vee r$ : Iyakaremye is not home or Iyakaremye is doing communal work

We know that by De Morgan's law,  $\sim(p \vee q) = \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$(\sim p) \wedge (\sim q)$
---	---	----------	----------	------------	------------------	----------------------------

T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Thus the negation of  $p \vee q$ : Murerwa is not home or Iyakaremye is doing communal work is  $\sim(p \vee q) = \sim p \wedge \sim q$

We see that  $\sim(p \vee q) = (\sim p) \wedge (\sim q)$

$\sim(p \vee q) = (\sim p) \wedge (\sim q)$ : Murerwa is home and Iyakaremye is not doing communal work.

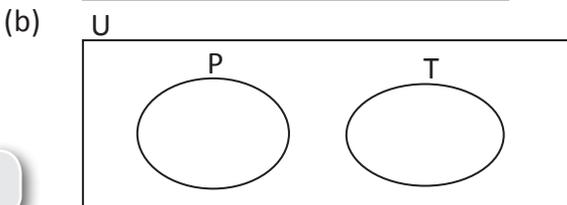
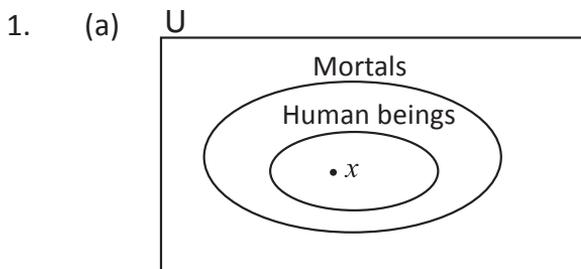
2. Let  $p$ : Nyirarukundo buys banana  
 $q$ : Muragijimana buys an orange  
 $p \vee q$ : If Nyirarukundo buys banana, then Muragijimana buys an orange.

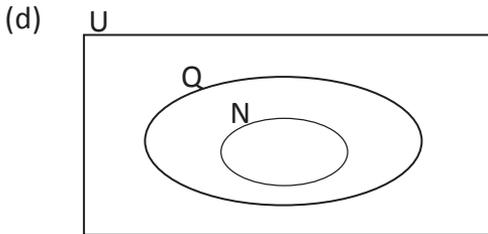
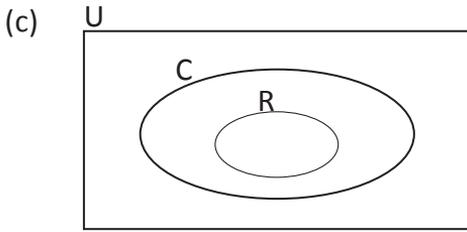
We know by De Morgan's Law that  $\sim p \Rightarrow q = (p \wedge \sim q)$

$p$	$q$	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim(p \Rightarrow q)$	$p \wedge (\sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	F	F

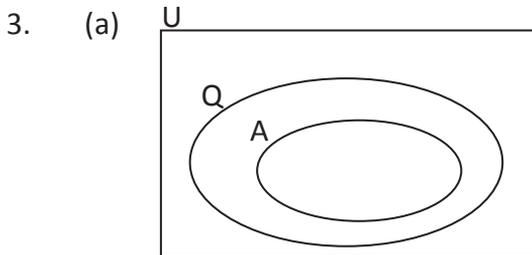
Thus, the negation of 'If Nyirarukundo buys a banana, then Muragijimana buys an orange' is 'Nyirarukundo buys an orange and Muragijimana does not buy an orange'.

**Task 2.5 on page 44**

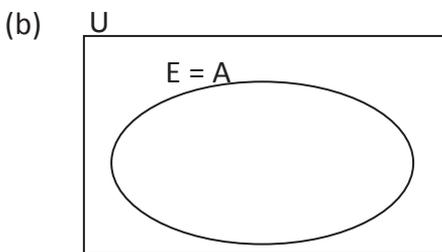




2. (a) Invalid  
 (b) Invalid  
 (c) Invalid  
 (d) Valid

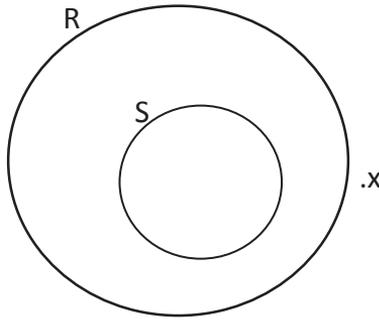


U = The set of all quadratic equations  
 Q = The set of all quadratic equations  
 A = The set of quadratic equations having two real roots



U = Set of all triangles  
 E = Set of all equilateral triangles  
 A = Set of all equiangular triangles [here sets A and E are the same].

4. (a)



The truth of the statement  $S_1$  is represented by placing the set  $S$  entirely inside the set  $R$ , as shown in the above figure.

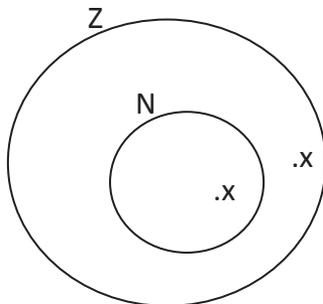
Now, the truth of the statement  $S_2$  is represented by placing labelled 'x' outside the set  $R$  as shown in figure.

Since the dot 'x' is outside the set  $R$  of rectangles, it is necessarily outside the set  $S$  of squares.

Therefore, we conclude that 'x' is not a square.

Hence the given argument is valid.

(b) Let  $Z$  = The set of integers and  $N$  = The set of natural numbers.



The truth of the statement  $S_1$  is represented by placing the set  $N$  entirely inside the set  $Z$  and the truth of the statement  $S_2$  is represented by placing a dot labeled 'x' inside the set  $Z$ . This dot may be inside  $N$  or outside it but inside  $Z$ .

Now  $S_1$  and  $S_2$  are true

The set  $N$  is entirely inside the  $Z$  and dot labeled 'x' may or may not be in  $N$

$x$  is not necessary a natural number

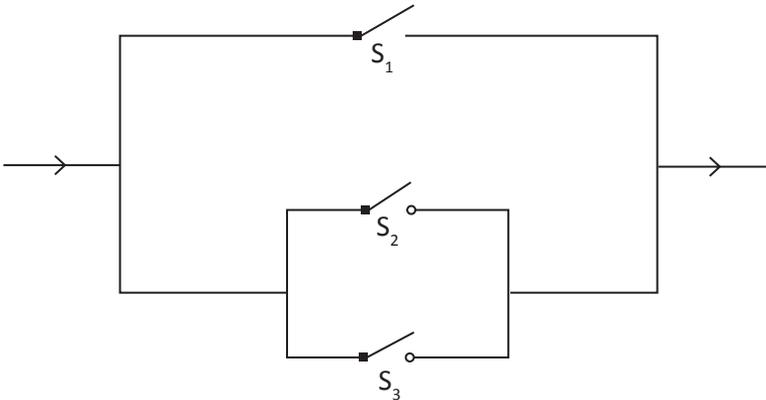
$S$  need not be true

Hence, the given argument is valid.

**Note:** An argument consisting of the hypothesis is  $S_1, S_2, \dots, S_n$  and conclusion  $S$  is said to be valid if  $S$  is true whenever all  $S_1, S_2, \dots, S_n$  are.

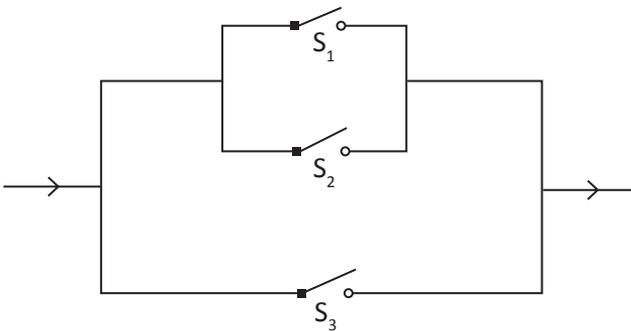
**Task 2.6 on page 46**

1. The circuit is given by the diagram below.



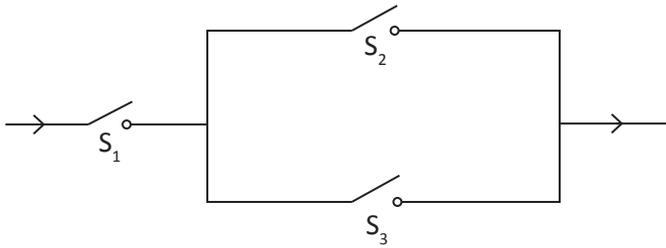
The current flows in the circuit if either  $S_1$  is closed or both  $S_1$  and  $S_3$  are closed.

2. The circuit is given by the diagram



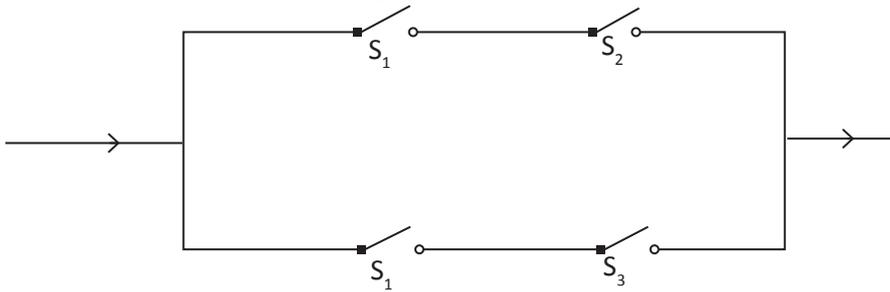
The current flows in the circuit if both  $S_1$  and  $S_2$  are closed or  $S_3$  is closed.

3. The circuit is given by the diagram below



There is a flow of current in the circuit if and only if  $S_1$  is closed and either  $S_2$  is closed or  $S_3$  is closed.

4. The circuit is given by the diagram below



There is a flow of current in the circuit if and only if  $S_1$  is closed and either  $S_2$  is closed or  $S_3$  is closed.

In other words, the above two circuits are equivalent.

So, we have  $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ .

# Topic area: Algebra

## Sub-topic area: Number and operations

UNIT  
**3**

### Binary operations

**Number of periods: 14**

#### **Learning objectives**

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define a group, a ring, an integral domain and a field</li><li>• Demonstrate that a set is (or is not) a group, a ring or a field under given operations</li><li>• Demonstrate that a subset of a group is (or is not) a sub group</li></ul>	<ul style="list-style-type: none"><li>• Determine the properties of a given binary operation</li><li>• Formulate, using adequate symbols, a property of a binary operation and its negation</li><li>• Construct the Cayley table of order 2, 3, 4</li><li>• Discover a mistake in an incorrect operation</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the importance and the use of properties of binary operations</li><li>• Show curiosity, patience, mutual respect and tolerance in the study of binary operations</li></ul>

#### **Content**

1. Definitions and properties
2. Groups and rings
3. Fields and integral domains
4. Cayley tables

## Materials required

Digital instruments such as calculators, counters

## Generic competences

- Problem-solving
- Cooperation
- Communication
- Lifelong learning

## Cross-cutting issues

### Peace and values education

Learners will be groups of various abilities and work with a respectful attitude.

### Standardization culture

Learners make use of rules and standards when constructing Cayley tables.

## Teaching and learning activities

### Introductory activity

Discuss in groups, patiently in mutual respect and tolerance, the main facts about binary operations. Let the students carry out Activity 3.1 on page 50 of the Student's Book.

### Main activities

1. Introduce the unit by giving some examples of binary operations.
2. Let them do Activity 3.2 on page 51 of the Student's Book. Then let one student from each group to present their findings. Guide them in defining the term group. Let them attempt, in pairs, Task 3.1 on page 53 of the Student's Book.
3. Guide them in defining rings. Let them attempt, in pairs, Task 3.2 on page 55 of the Student's Book.
4. Guide them in defining fields and integral domains. Explain what Cayley tables are and demonstrate to them how to construct tables of order 2, 3, 4.

5. Guide them on how to discover a mistake in an incorrect operation. Let them, in pairs, attempt Task 3.3 on page 60 of the Student's Book.

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them solve additional problems on construction of Cayley tables and allow them to do extra exercises. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

### (b) Slow learners

You should provide them with the non-restrictive environment that provides and maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

## Additional tasks

### Task 3A - For slow learners

1. Let  $*$  be a binary operation defined on the set  $\mathbb{Z}$  of all integers by  $x * y = x + y - 1$ .
- Determine whether the operation is commutative
  - Find an identity element
  - Find a symmetric element

### Answers

1. a) The operation is commutative if  $x, y \in \mathbb{Z}$ ;  $x * y = y * x$   
 $x * y = x + y - 1$  and  $y * x = y + x - 1 = x + y - 1$   
Thus  $x * y = y * x$  and the operation is commutative
- b) There is an identity element  $e \in \mathbb{Z}$  if  $\forall x \in \mathbb{Z}; x * e = x$   
 $x * e = x + e - 1 = x \Leftrightarrow e - 1 = 0 \Leftrightarrow e = 1$ .

Thus the  $*$  operation admits  $e = 1$  as the identity element in  $\mathbb{Z}$ .

c) A symmetric (inverse) element of  $x$  in  $\mathbb{Z}$  is  $x'$  if  $\forall x \in \mathbb{Z}; x * x' = e$   
 $x * x' = x + x' - 1 = e \Leftrightarrow x' = e - x + 1 = 1 - x + 1 = 2 - x$

The  $*$  operation admits  $x' = 2 - x$  as a symmetric (inverse) of  $x$  in  $\mathbb{Z}$ .

### Task 3B - For talented learners

1. Let  $*$  be a binary operation defined on the set  $G$  by  $a * b = a + b + ab$ . Show that  $(G, *)$  is a commutative group

#### Answers

1. The set  $G$  with the operation  $*$  written  $(G, *)$  is commutative group since it satisfies the following properties:

1) Closure:  $\forall a, b \in G: a * b = [(a + b) + (ab)] \in G$  Verified

2) Commutative:  $\forall a, b \in G: a * b = b * a$

In fact  $a * b = a + b + ab$  and  $b * a = b + a + ba = a + b + ab$

Thus,  $a * b = b * a$ . Verified

3) Identity element:  $\forall a \in G, \exists e \in G: a * e = e * a = a$ . We find the identity element,

$$a * e = a + e + ae = a \Leftrightarrow e + ae = 0 \Leftrightarrow e(1 + a) = 0 \Leftrightarrow e = 0$$

The identity element is  $e = 0$ . Verified

4) Inverse element:  $\forall a \in G, \exists a' \in G: a * a' = a' * a = 0$

$$a * a' = a + a' + aa' = 0 \Leftrightarrow a + a'(1 + a) = 0 \Leftrightarrow a'(1 + a) = -a$$

$$a' = \frac{-a}{1 + a}$$

Thus, the inverse of  $a$  is  $a' = \frac{-a}{1 + a}$ . Verified

5) Commutative:  $\forall a, b \in G: a * b = b * a$

In fact  $a * b = a + b + ab$  and  $b * a = b + a + ba = a + b + ab$

Thus,  $a * b = b * a$ . Verified

All the 5 properties above prove that  $(G, *)$  is a commutative group.

## Assessment criteria

1. Explain why grouping, interchanging and distributing is correct or not, depending on context.
2. Carry out binary operations and determine their properties.

## Answers to Tasks of Unit 3 in the Student's Book

### Task 3.1 on page 53

1. We show if the following properties hold.
  - The sum of any two real numbers is a real number, so the real numbers are closed under addition.  
 $\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$
  - Addition on the real numbers is associative.  
 $\forall x, y, z \in \mathbb{R}; (x + y) + z = x + (y + z) \in \mathbb{R}$
  - The identity element is 0, since  
 $\forall x \in \mathbb{R}; x + 0 = 0 + x = x$
  - The inverse of  $x$  is  $-x$  for all  $x \in \mathbb{R}$ , since  
 $x + (-x) = (-x) + x = 0$
  - Addition of real numbers is commutative.  
 $\forall x, y \in \mathbb{R}; x + y = y + x \in \mathbb{R}$

Conclusion:  $\mathbb{R}$  is a commutative group ( Abelian Group)

2. We show if the following properties hold.
  - For any numbers  $a, b \in \mathbb{Z}; a + b \in \mathbb{Z}$  (closure)
  - For all  $a, b, c \in \mathbb{Z}; (x + b) + c = a + (b + c)$  (associative)
  - For every  $a \in \mathbb{Z}, a + 0 = 0 + a = a$  (0 is the identity element)
  - For every  $a \in \mathbb{Z}; a + (-a) = -a + a = 0$  ( $-a$  is the inverse element of  $a$ )Since all the properties are satisfied, then it is a group under addition.
3. (a) Odd integers under addition is not a group  
(b)  $\mathbb{Z}$  is not a group under multiplication  
(c)  $\mathbb{N}$  is not a group under addition  
(d)  $\mathbb{R}$  is a group under multiplication  
(e)  $\mathbb{Q}$  is a group under addition

### Task 3.2 on page 55

These are obvious as they are from the following basic properties of a commutative ring

1. If  $(\mathbb{R}, +, \cdot)$  is a commutative ring, with  $a, b, c \in \mathbb{R}$ . The following hold:

$$a \cdot 0 = 0 \cdot a = 0$$

$$a(-b) = (-a)b = -ab$$

$$-(-a) = a$$

$$\text{If } a + b = a + c \text{ then } b = c$$

$$\text{If } a + a = a \text{ then } a = 0$$

$$(-1)a = -a$$

2. Assume in addition that  $\mathbb{R}$  has an identity 1, then

$$(-1)a = -a$$

$$\text{If } a \in \mathbb{R} \text{ has a multiplicative identity } a^{-1} \text{ then } ab = 0 \implies b = 0.$$

### Task 3.3 on page 60

1. (a) Commutativity  
 (b) The identity element  
 (c) The inverse element
2. The Cayley table for the set  $S = \{1, -1, i, -i\}$  under “ $\cdot$ ”

.	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

- (i) All the results are elements of the set  $S = \{1, -1, i, -i\}$ . The operation is **closure** on the set  $S$ .
- (ii) There is **associativity**: For instance,  $i \cdot (1 \cdot -i) = (i \cdot 1) \cdot -i = 1$
- (iii) The **identity element** is 1 since For every element  $x$  of  $S$ ,  
 $x \cdot 1 = x \cdot 1 = x$
- (iv) There exists the **inverse element**: for every element  $x$  of  $S$ : For instance the inverse of 1 is  $-1$ , the inverse of  $i$  is  $-i$ .
- (v) There is a **commutativity**: For instance  $-1 \cdot i = i \cdot -1$  (The entry

inside the table are symmetrical about the leading diagonal.)

We conclude that  $(S, \bullet)$  is a commutative group.

3. Cayley table for the set  $S = \{f, g, h\}$  under the composition operation ' $\circ$ '

$\circ$	f	g	h
f	f	g	h
g	g	f	-h
h	h	-h	f

- (a) The composition is not closure since the result  $-h$  is not the element of the original set.
- (b) The composition is not associative since for example  
 $(f \circ g) \circ h = g \circ h = -h$  and  $f \circ (g \circ h) = f \circ (-h) = -h$   
Which shows us that  $(f \circ g) \circ h = f \circ (g \circ h)$
- (c) f is the identity element.
- (d) Each function is the inverse of itself,  $g \circ h = h \circ g = -h$ .
- (e) The composition is commutative since, for instance,  $f \circ g = g \circ f = g$

# Topic area: Algebra

## Sub-topic area: Numbers and operations

UNIT

# 4

## Set $\mathbb{R}$ of real numbers

**Number of periods: 24**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Match a number and the set to which it belongs</li><li>• Define a power, an exponential, a radical, a logarithm, the absolute value of a real number.</li></ul>	<ul style="list-style-type: none"><li>• Classify numbers into naturals, integers, rational and irrationals</li><li>• Determine the restrictions on the variables in rational and irrational expressions</li><li>• Illustrate each property of a power, an exponential, a radical, a logarithm and the absolute value of a real number</li><li>• Use logarithm and exponentials to model simple problems about growth, decay, compound interest and magnitude of an earthquake.</li><li>• Transform a logarithmic expression to equivalent power or radical form, and vice versa</li><li>• Rewrite an expression containing “absolute value” using order relation</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the importance and the use of properties of operations on real numbers</li><li>• Show curiosity for the study of operations on real numbers</li></ul>

## Content

1. Properties of real numbers
2. Absolute value and its properties
3. Powers and radicals
4. Decimal logarithms properties

## Materials required

Graph papers, manila papers, digital technology instruments including calculators

## Generic competences

- Communication
- Problem-solving
- Research
- Cooperation
- Critical thinking

## Cross-cutting issues

- **Peace and values education**  
Listening to others and contributing to solutions of challenges.
- **Inclusive education**  
For learners with impaired vision, fellow student's work with them in groups using large teaching learning visual aids. (See Activity 4.2.)
- **Financial education**  
Tasks and activities such Task 4.5 that involve bank transactions and investments.

## Teaching and learning activities

### Introductory activity

Guide students to carry out research in advance in the library about sets of numbers (natural numbers, integers, rational numbers and irrational numbers). This is outlined in Activity 4.1 on page 63 of the Student's Book.

Give them a mental task to find out the main facts about sets of real numbers.

### **Main activities**

1. Introduce the unit by giving some examples of sets of numbers.
2. Guide them in giving the definitions of a power, an exponential, a radical, a logarithm and the absolute value of a real number. Guide them in classifying numbers.
3. Let students work on Activity 4.2 on page 65 of the Student's Book.
4. Let them attempt Activity 4.3 on page 66 to find out the meaning of absolute value of a real number. This should be followed by Task 4.1 on page 68 to grasp the concept of absolute values.
5. Let them research on the meaning of the term 'power' in pairs. They will do Activity 4.4 on page 68 of the Student's Book.
6. Let them do Activity 4.5 on page 71 that involves researching on the meaning of radicals. You will supervise the class discussion and help them come up with a concise meaning of radicals.
7. Take them through the symbolisms and meanings of roots and surds. Guide them to illustrate each property of a power, an exponential, a radical, a logarithm and an absolute value of a real number. Explain the meaning of rationalization. Let them work on Task 4.2 on page 73.
8. Guide them in using logarithms and exponentials to model simple problems such as on growth, decay, compound interest and magnitude of an earthquake. Give them the Activity 4.6 on page 75, Task 4.3 on page 76 and Activity 4.7 on page 76 of the Student's Book to practise calculations involving logarithms. The Task 4.4 on page 78 should reinforce this.
9. The Task 4.5 on pages 79 and 80 of the Student's Book should be apt for practise on applications.

## **Learners of varying strengths and abilities**

### **(a) Gifted and talented**

You can provide more advanced material to the learners. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

**(b) Slow learners**

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

**(c) Intellectual impairment**

Try to understand the specific talents of the learner and develop them. Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task . Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

## Additional tasks

### **Task 4A - For slow learners**

1. Solve the system of equations

$$2^{x-1} + 2^{1-x} = 2$$

2. Simplify the following

i)  $7\sqrt{2} - \sqrt{12} + \sqrt{48}$

ii)  $\sqrt{45} - \sqrt{20} + \sqrt{80}$

iii)  $\sqrt{50} - 3\sqrt{18} + 2\sqrt{8} - 7\sqrt{2}$

### **Answers**

1.  $2^{x-1} + 2^{1-x} = 2$

$$\frac{2^x}{2} + \frac{2}{2^x} = 2$$

$$\frac{2^{2x} + 4}{2 \times 2^x} = 2$$

$$2^{2x} + 4 = 4 \times 2^x$$

$$2^{2x} - 4 \times 2^x + 4 = 0$$

Let  $y = 2^x$ , the equation above becomes

$$y^2 - 4y + 4 = 0$$

$$(y - 2)^2 = 0$$

$$y = 2$$

$$2^x = 2$$

$$x = 1$$

$$\begin{aligned} 2. \quad \text{i)} \quad 7\sqrt{3} - \sqrt{12} + \sqrt{48} &= 7\sqrt{3} - \sqrt{4 \times 3} + \sqrt{16 \times 3} \\ &= 7\sqrt{3} - \sqrt{4} \times \sqrt{3} + \sqrt{16} \times \sqrt{3} \\ &= 7\sqrt{3} - 2\sqrt{3} + 4\sqrt{3} = 9\sqrt{3} - \end{aligned}$$

$$\begin{aligned} \text{ii)} \quad \sqrt{45} - \sqrt{20} - \sqrt{80} &= \sqrt{9 \times 5} - \sqrt{4 \times 5} - \sqrt{16 \times 5} \\ &= \sqrt{9} \sqrt{5} - \sqrt{4} \sqrt{5} - \sqrt{16} \sqrt{5} \\ &= 3\sqrt{5} - 2\sqrt{5} + 4\sqrt{5} = -3\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{iii)} \quad \sqrt{50} + 3\sqrt{18} - 2\sqrt{8} - 7\sqrt{2} &= \sqrt{25 \times 2} + 3\sqrt{9 \times 2} - 2\sqrt{4 \times 2} - 7\sqrt{2} \\ &= \sqrt{25} \sqrt{2} + 3\sqrt{9} \sqrt{2} - 2\sqrt{4} \sqrt{2} - 7\sqrt{2} \\ &= 5\sqrt{2} + 9\sqrt{2} - 4\sqrt{2} - 7\sqrt{2} = 3\sqrt{2} \end{aligned}$$

### **Task 4B - For talented learners**

1. A species of lions is introduced into Akagera National Park where previously there were no lions. Six lions were introduced in 2015. It is expected that the population will increase according to  $L_t = L_0 \times 2^{0.4t}$  where  $t$  is the time since the introduction.
- Find  $L_0$
  - Find the expected lion population in 2020
  - In which year will the number of lions be 100.

### **Answers**

1. a)  $L_0 = 6$  lions

b)  $L_t = L_0 \times 2^{0.4t}$

$$L_5 = 6 \times 2^{0.4 \times 5}$$

$$L_5 = 6 \times 2^{0.4 \times 5} = 6 \times 2^2 = 24$$

The expected number of lions in 2020 is **24 lions**

c)  $L_t = L_0 \times 2^{0.4t}$

$$100 = 6 \times 2^{0.4t}$$

$$\frac{100}{6} = 2^{0.4t}$$

$$\log \frac{50}{3} = \log 2^{0.4t}$$

$$\log 50 - \log 3 = 0.4t \log 2$$

$$t = \frac{\log 50 - \log 3}{0.4 \log 2}$$

$$t = \frac{2.813}{0.277} = 10.15$$

The population lion will be 100 in ten years ahead i.e. in 2025.

## Assessment criteria

Use mathematical logic to understand and perform operations on the set of real numbers and its subsets using the properties of algebraic structures

## Answers to Tasks of Unit 4 in the Student's Book

### Task 4.1 on page 68

- |  |                               |
|--|-------------------------------|
| (a) $x = \left\{ \frac{5}{2}, \frac{15}{2} \right\}$ | (b) $x \in [-3, 3]$           |
| (c) $x \leq -5$ or $x \geq -2$                       | (d) $x = \{-1, 4\}$           |
| (e) $x < -3$ or $x > -1$                             | (f) $x \in ]-\frac{7}{2}, 4[$ |
| (g) $x \in \mathbb{R}$                               | (h) $x \geq 4$                |

### Task 4.2 on page 73

- |  |  |
|--|--|
| 1. $\frac{3\sqrt{2}}{2}$               | 2. $\frac{1\sqrt{7}}{7}$                 |
| 3. $\frac{2}{11}$                      | 4. $\frac{3\sqrt{10}}{5}$                |
| 5. $\frac{1\sqrt{3}}{9}$               | 6. $\frac{1\sqrt{2}}{2}$                 |
| 7. $\sqrt{2} + 1$                      | 8. $\frac{\sqrt{3}}{23}(15 - 3\sqrt{2})$ |
| 9. $\frac{1}{3}(4\sqrt{3} + 6)$        | 10. $-5(2 + \sqrt{5})$                   |
| 11. $\frac{1}{4}(\sqrt{7} + \sqrt{3})$ | 12. $4(2 + \sqrt{3})$                    |
| 13. $\sqrt{5} - 2$                     | 14. $\frac{1}{3}(7\sqrt{3} + 2)$         |
| 15. $3 + \sqrt{5}$                     | 16. $3(\sqrt{3} + \sqrt{2})$             |

17.  $\frac{3}{19}(10 - \sqrt{3})$

18.  $3 + 2\sqrt{2}$

19.  $\frac{2}{3}(7 - 2\sqrt{7})$

20.  $\frac{1}{2}(1 + \sqrt{5})$

21.  $\frac{1}{4}(\sqrt{11} + \sqrt{7})$

22.  $\frac{1}{6}(9 + \sqrt{3})$

23.  $\frac{1}{14}(9\sqrt{2} - 20)$

24.  $\frac{1}{6}(3\sqrt{2} + 2\sqrt{3})$

25.  $\frac{1}{2}(2 + \sqrt{2})$

**Task 4.3 on page 76**

(a) 6

(b) 1

(c) -3

(d) 0

(e)  $\frac{1}{2}$

(f)  $\frac{1}{4}$

(g)  $-\frac{1}{4}$

(h)  $\frac{5}{2}$

(i)  $\frac{2}{3}$

(j)  $\frac{1}{2}$

(k)  $\frac{10}{3}$

(l)  $\frac{3}{2}$

(m) 2a

(n) a + 3

(o) a - 1

(p) a - b

**Task 4.4 on page 78**

1. (a)  $\log 30$

(b)  $\log 8$

(c)  $\log 4$

(d)  $\log 35$

(e)  $\log 4$

(f)  $\log 105$

(g)  $\log 90$

(h)  $\log 2$

(i)  $\log 16$

(j)  $\log 6,000$

(k)  $\log 4$

(l)  $\log 2$

(m)  $\log 0.005$

(n)  $\log 20$

(o)  $\log 28$

2. (a)  $\log 32$

(b)  $\log 256$

(c)  $3 - \log 2$

(d)  $2 - \log 4$

(e)  $\log 14$

(f)  $\log 2$

(g) 1

(h) 8

(i) 1

3. (a) 2

(b)  $\frac{3}{2}$

(c) 4

(d)  $\frac{3}{2}$

(e)  $-\frac{5}{2}$

(f)  $-\frac{3}{4}$

**Task 4.5 on page 79**

1. (a) The growth factor for the town is 1.024.

(b)  $P(t) = 35,000(1.024)^t$ .

- (c) Since the equation is designed for future, we cannot estimate the population of the town in the year 2007.
2. (a) The growth factor for the investment is 1.05.  
(b)  $A(t) = 300,000(1.05)^t$  where  $A(t)$ : amount of money after  $t$  years,  $t$ : period of  $t$  years.  
(c) The money will double in between 14 and 15 years.
3. (a) The decay factor for the value of the car is 0.85.  
(b)  $C(t) = 25,000(0.85)^t$  where  $C(t)$ : the value of the car after  $t$  years,  $t$ : period of  $t$  years.  
(c) In 10 years the car will be worth 4921.86 FRW.
4. The amount of investment at the end of 5 years is 819,308.22 FRW.
5. At the end of 96 minutes the number of bacteria would be 602,248.76.

# Topic area: Algebra

## Sub-topic area: Equation and inequalities

UNIT

# 5

## Linear equations and inequalities

**Number of periods: 12**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>List and clarify the steps in modeling a problem by linear equations and inequalities</li></ul>	<ul style="list-style-type: none"><li>Equations and inequalities in one unknown</li><li>Parametric equations and inequalities in one unknown.</li><li>Simultaneous equations in two unknowns.</li><li>Applications:<ul style="list-style-type: none"><li>Economics (problems about supply and demand analysis)</li><li>Physics (linear motions, electric circuits)</li><li>Chemistry (balancing equations)</li></ul></li></ul>	<ul style="list-style-type: none"><li>Appreciate, value and care for situations involving two linear equations and linear inequalities in daily life situation</li><li>Show curiosity about linear equations and linear inequalities</li></ul>

### Content

- Equations and inequalities in one unknown
- Parametric equations and inequalities
- Simultaneous equations in two unknowns

4. Applications
  - Economics
  - Physics
  - Chemistry

## Materials required

Geometric instruments (ruler, T-square ...), calculators.

## Competences

- Communication
- Research
- Cooperation
- Problem solving
- Critical thinking

## Cross-cutting issues

- **Financial education**

To plan about how to use money and to do savings.

- **Inclusive education**

Those with impairments and disabilities are mixed with the rest in the groups and assisted when necessary.

## Teaching and learning activities

### Introductory activity

Let students discuss in groups the importance and necessity of linear equations and inequalities.

### Main activities

1. Let students work on Activity 5.1 on page 82 of the Student's Book. Encourage them to work with cooperation when it comes to discussions.
2. Introduce the unit by listing and clarifying the steps in modelling problems of linear equations and inequalities.
3. Let them attempt Activity 5.2 and Task 5.1 on linear equations. These are on pages 84 and 85 respectively. They should practise until they perfect the solving of equations.

4. Let them do Activity 5.3 on page 85 of the Student's Book of the student's to introduce them to linear inequalities. Ensure they differentiate them from linear equations. They should be able to appreciate the differences.
5. Introduce them to graphs of inequalities. Let them tackle Task 5.2 on page 88 of the Student's Book. They can do the first two numbers in pairs before working on the rest as individuals.
6. Introduce parametric equations by asking learners to research on them. Let them present their findings in class as is recommended in Activity 5.4 on page 88 of the Student's Book.
7. Let them work on Activity 5.5 on page 90 in groups. They should take turns in calculations and drawing of graphs. The graphs can be drawn on large manila papers to enable those with visual impairment participate fully.
8. Introduce simultaneous equations using the mental task on page 91. Give them adequate practice on finding solutions using different methods.
9. Introduce the applications of equations and inequalities. Let learners discuss and express their appreciation of these in real life, in areas of trade, linear motions, electric circuits and balancing equations. Let them attempt the Task 5.3, Activity 5.6 and Task 5.4 on pages 93 94 and 96 of the Student's Book.

## **Reinforcement activity**

Solving linear equations and simultaneous equations on a graph paper.

## **Learners of varying strengths and abilities**

### **(a) Gifted and talented**

Guide the learners to study units ahead of others. You can provide more advanced material to the learner.

You can ask them to research further on application of equations and inequalities in real life and to give their presentations in class.

### **(b) Slow learners**

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

## Additional tasks

### Task 5A - For slow learners

1. Solve and graph the solution of the following inequality

$$\frac{3(x-2)}{5} < \frac{5(2-x)}{3}$$

### Answers

1. 
$$\frac{3(x-2)}{5} < \frac{5(2-x)}{3}$$

$$\frac{3(x-2)}{5} < \frac{10-5x}{3}$$

$$\frac{3(3x-6)}{15} < \frac{5(10-5x)}{15}$$

$$9x - 18 < 50 - 25x$$

$$34x < 68$$

$$x < 2$$

The solution set is

The solution can be graphed on the number line as follows:



### Task 5B - For talented learners

A solution is to be kept between 77°F and 86°F. What is the range in temperature in degree Celcius (C) if the Celcius-Fahrenheit (F) conversion formula is given by

$$F = \frac{9}{5} C + 32?$$

### Answer

We know that  $F = \frac{9}{5} C + 32$ ?

It is given that  $77 < F < 86$

$$77 < \frac{9}{5} C + 32 < 86$$

$$77 - 32 < \frac{9}{5} C < 86 - 32$$

$$45 < \frac{9}{5} C < 54$$

$$\frac{5}{9} C \times 45 < C < \frac{5}{9} \times 54$$

$$25 < C < 30$$

Thus, the required range of temperature in °C is between 25°C and 30°C.

## Assessment criteria

Model and solve algebraically or graphically daily life's problems using linear equations or inequalities.

## Answers to Tasks of Unit 5 in the Student's Book

### Task 5.1 on page 85

(a)  $(x + 2)(2x - 1) = 0$

$$x + 2 = 0 \text{ or } 2x - 1 = 0$$

$$x = -2 \text{ or } x = \frac{1}{2}$$

(b)  $(5x - 15)(3x - 9) = 0$

$$5x - 15 = 0 \text{ or } 3x - 9 = 0$$

$$x = \frac{15}{5} \text{ or } x = \frac{9}{3}$$

$$x = 3 \text{ or } x = 3$$

$$x = 3$$

(c)  $x^2 - 5x = 0$

$$x(x - 5) = 0$$

$$x = 0 \text{ or } x - 5 = 0$$

$$x = 0 \text{ or } x = 5$$

(d)  $\frac{3x - 6}{x + 1} = 0$

$$3x = 6 \text{ and } x + 1 \neq 0$$

$$3x = 6 \text{ and } x \neq -1$$

$$x = 2 \text{ and } x \neq -1$$

(e)  $\frac{x - 2}{5x + 3} = 0$

$$x - 2 = 0 \text{ and } 5x + 3 \neq 0$$

$$x = 2 \text{ and } x \neq -\frac{3}{5}$$

(f)  $3(x + 7) = 0$

$$3x + 21 = 0$$

$$x = -7$$

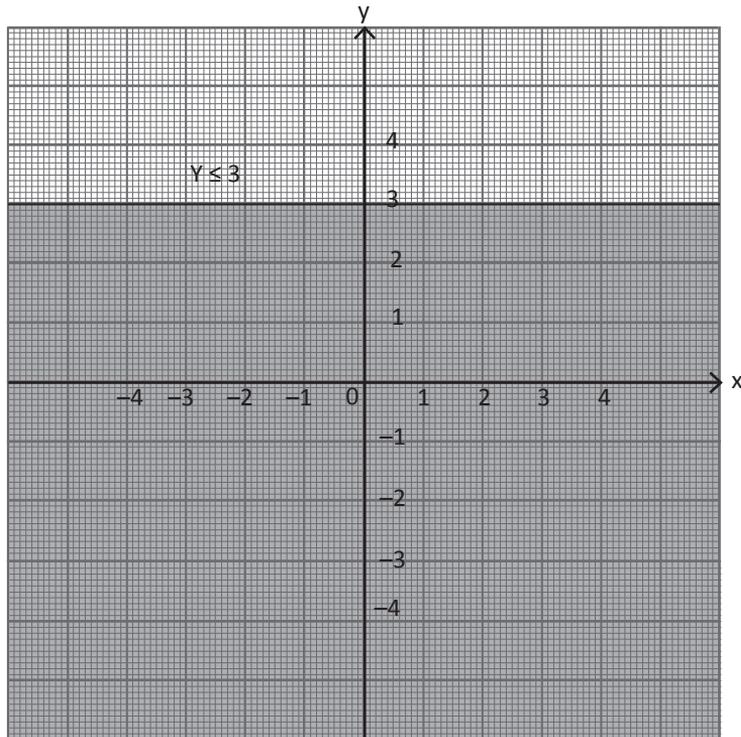
(g)  $\frac{3-x}{2x-7} = 0$

$$3 - x = 0 \text{ and } 2x - 7 \neq 0$$

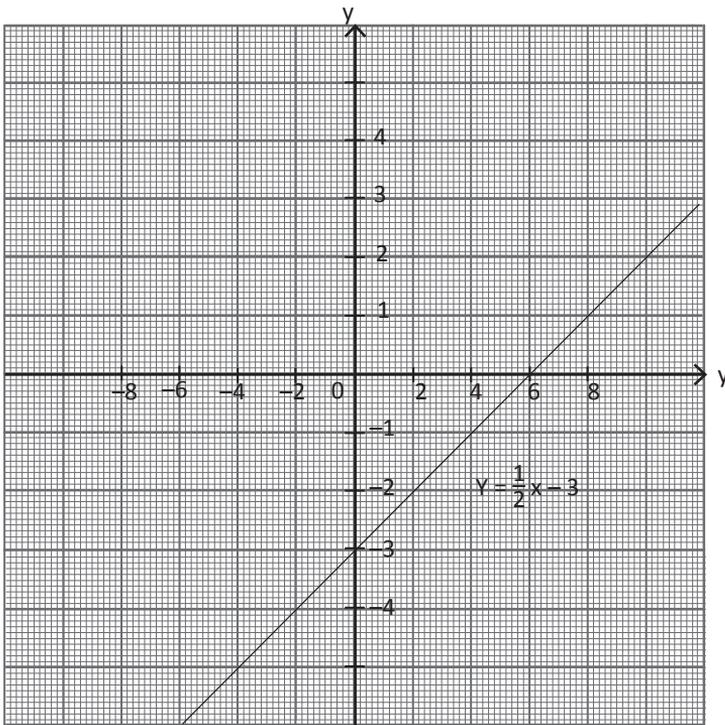
$$x = 3 \text{ where } x \neq \frac{7}{2}$$

**Task 5.2 on page 88**

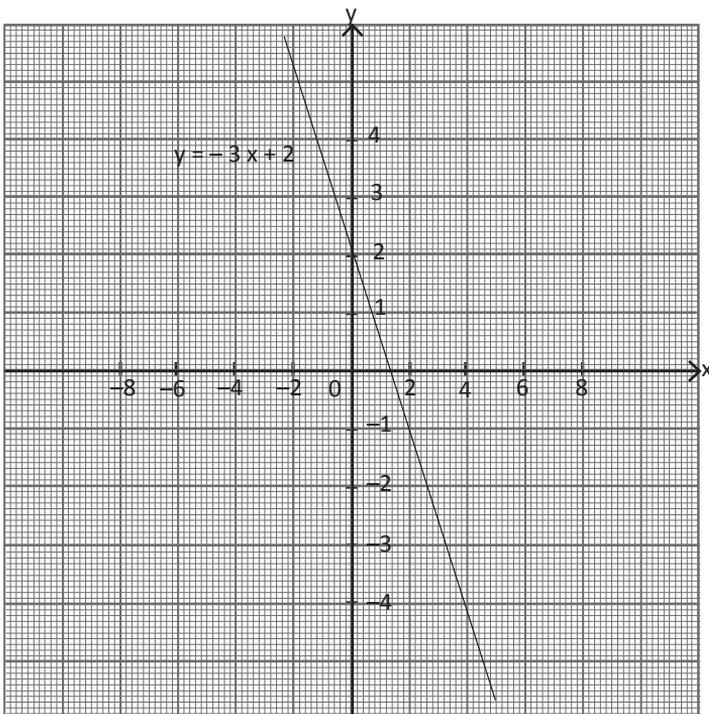
1.



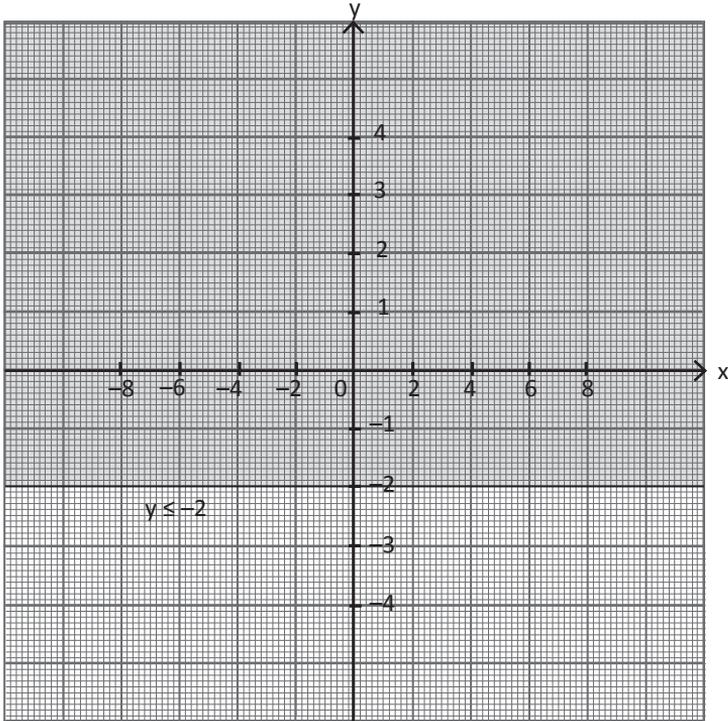
2.



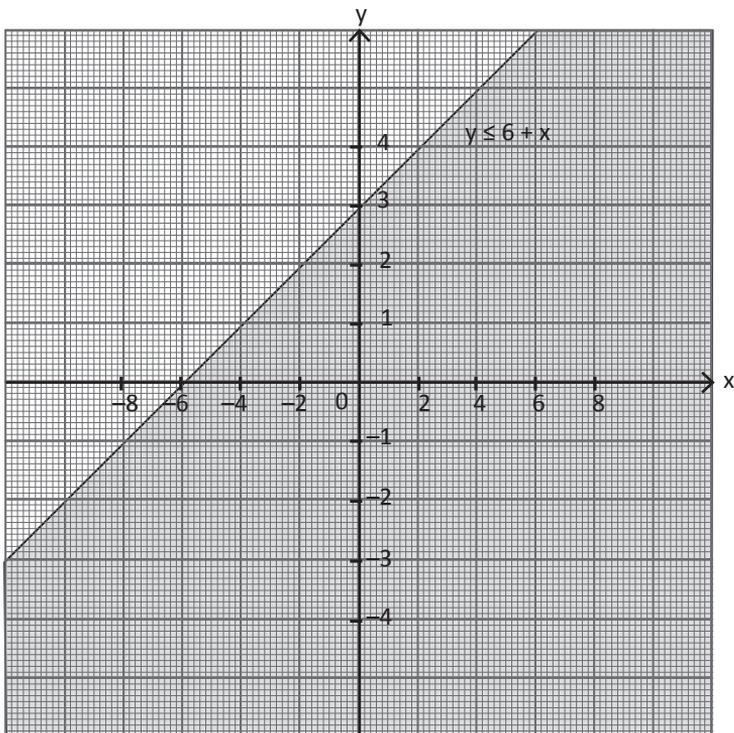
3.



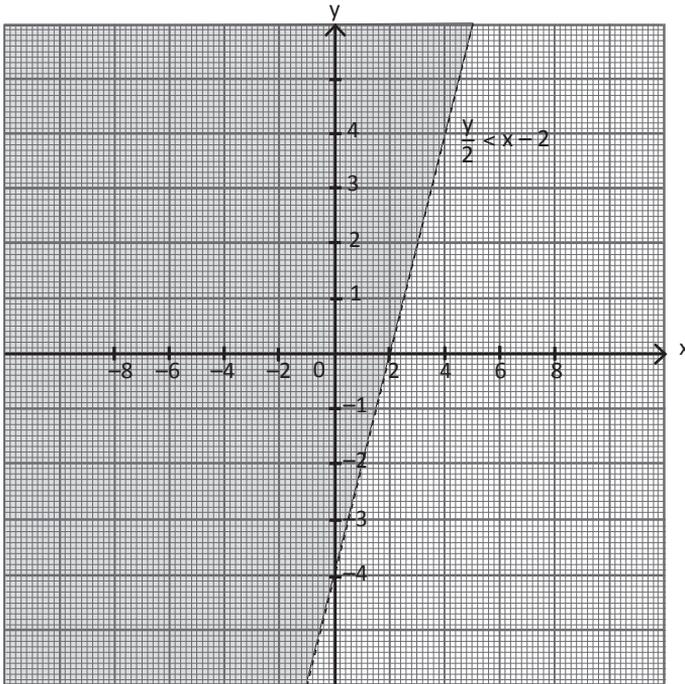
4.



5.



6.



**Task 5.3 on page 93**

1. Fish fillet cost  $x$  FRW  
Egg roll cost  $y$  FRW

We form the system  $\begin{cases} x = y + 500 \\ x + y = 2,000 \end{cases}$

We solve for  $x$  and  $y$

$$\begin{cases} x = y + 500 \\ y + 500 + y = 2,000 \end{cases}$$

$$\begin{cases} x = y + 500 \\ y + y = 1,500 \end{cases}$$

$$\begin{cases} x = y + 500 \\ 2y = 1,500 \end{cases}$$

$$\begin{cases} x = y + 500 \\ y = 750 \end{cases}$$

$$\begin{cases} x = 1,250 \\ y = 750 \end{cases}$$

- (a) The unit price of fish fillet is 1,250 FRW.
- (b) The unit price of egg roll is 750 FRW.

2. If  $x$  crates of Fanta and  $y$  crates of Coke are distributed

We form a system of equations

$$\begin{cases} x + 100 = y \\ x + y = 400 \end{cases}$$

We solve for  $x$  and  $y$

$$\begin{cases} x - y = -100 \\ x + y = 400 \end{cases}$$

$$\begin{cases} x - y = -100 \\ 2x = 300 \end{cases}$$

$$\begin{cases} x - y = -100 \\ x = 150 \end{cases}$$

$$\begin{cases} y = 150 + 100 \\ x = 150 \end{cases}$$

$$\begin{cases} y = 250 \\ x = 150 \end{cases}$$

150 crates of Fanta and 250 crates of Coke are distributed

3. A box file cost  $x$  FRW and a file folder cost  $y$  FRW.

We form the system of equations as:

$$\begin{cases} 3x = 2y + 10,000 \\ x + 2y = 5,000 \end{cases}$$

We solve for  $x$  and  $y$

$$\begin{cases} 3x = 2y + 10,000 \\ -x - 2y = -5,000 \end{cases}$$

$$\begin{cases} 2x = 5,000 \\ 2y = 5,000 - x \end{cases}$$

$$\begin{cases} x = 2,500 \\ y = 1,250 \end{cases}$$

Rutaya spends

(i) 2,500 FRW on a box file and

(ii) 1,250 FRW on a file folder.

4. If  $x_1$  is the number of trips on transporting sand to site 1  
 $y_1$  is the number of trips on transporting gravel to site 1  
If  $x_2$  is the number of trips on transporting sand to site 2  
 $y_2$  is the number of trips on transporting gravel to site 2

We form the system as below:

$$\begin{cases} 6,500x_1 + 10,000y_1 = 63,000 \\ 10,000x_2 + 13,000y_2 = 90,000 \end{cases}$$

Since we have the system of two equations with 4 unknowns, we cannot solve the system and thus there is no solution for the problem.

5. If  $x$  is the number of hard-covered books and  $y$  is the number of paperback

Time available for binding is 500 hours = 30,000 min

We form the system of equation and solve:

$$\begin{cases} 3x + 2y = 30,000 \\ x + y = 12,000 \end{cases}$$

$$\begin{cases} 3x + 2y = 30,000 \\ -2x + 2y = -24,000 \end{cases}$$

$$\begin{cases} x = 6000 \\ y = -12,000 - x \end{cases}$$

$$\begin{cases} x = 6,000 \\ y = 6,000 \end{cases}$$

The number of hard-covered books is 6,000 and the number of paperbacks is 6,000.

**Task 5.4 on page 96**

1. 68,260.87
2. 2h 56min 28sec
3.  $7,500d + 3,000c \geq 25,500$

# Topic area: Algebra

## Sub-topic area: Equations and inequalities

UNIT

# 6

## Quadratic equation and inequalities

**Number of periods: 18**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define a quadratic equation</li><li>• Be able to solve problems related to quadratic equations</li></ul>	<ul style="list-style-type: none"><li>• Apply critical thinking by solving any situation related to quadratic equations (economics problems, finance problems.)</li><li>• Determine the sign, sum and product of a quadratic equation</li><li>• Discuss the parameter of a quadratic equation and a quadratic inequality</li><li>• Explain how to reduce bi-quadratic equations to a quadratic equation and other equations of degrees greater than 2</li></ul>	<ul style="list-style-type: none"><li>• Appreciate value and care for situations involving to quadratic equations and quadratic inequalities in daily life situation.</li><li>• Show curiosity about quadratic equations and quadratic inequalities.</li></ul>

### Content

1. Equations in one unknown
2. Inequalities in one unknown

3. Simultaneous equations in two unknowns
4. Applications: Physics  
Masonry

## Materials required

Geometrical instruments (ruler, T-square), calculators.

## Generic competences

- Problem-solving
- Cooperation
- Communication
- Lifelong learning

## Cross-cutting issues

- **Inclusive education**

Inclusion of learners with disabilities in group work.

- **Standardisation culture**

Accuracy in calculations and drawing of graphs.

- **Financial education**

Examples on buying and selling, profit and loss.

## Teaching and learning activities

### Introductory activity

Guide students to discuss in groups the importance and necessity of quadratic equations and inequalities and how it takes the place in finance, economics and physics, for example. Let them do Activity 6.1 on page 98 of the Student's Book. This involves researching on the definition of a quadratic equation.

### Main activities

1. Introduce the unit by giving some examples of quadratic equations.
2. Give them concise the definition of a quadratic equation after listening to their research findings.

3. Help the students to solve problems related to quadratic equations. Let them attempt Task 6.1 on page 99 of the Student's Book.
4. Guide them to be able to solve equations by looking for the sign, sum and product of a quadratic equation. Task 6.2 on page 101 involves solving equations by use of sum and product. Let them work in pairs to tackle Activity 6.2 on page 101.
5. Introduce to them solving of quadratic equations by factorisation. Let them attempt Task 6.3 on page 102 of the Student's Book.
6. Introduce inequalities in one unknown and the use of sign diagrams. Let them attempt Task 6.4 on page 103, Task 6.5 on page 104 and Task 6.6 on page 105.
7. Encourage them to define what parametric equations are. Guide them in solving parametric equations. Let them attempt Tasks 6.7 and Task 6.8 on pages 106 and 107 respectively of the Student's Book.
8. Guide learners in researching on and discussing the applications of quadratic equations in real life. This is brought out in Activity 6.3 on page 108. Assist them to appreciate this importance. Let them attempt Task 6.9 on page 111 of the Student's Book.

### **Reinforcement activity**

Sketching on graph paper the graphs of quadratic equations

### **Learners of varying strengths and abilities**

#### **(a) Gifted and talented**

You can provide more advanced material to the learner. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

#### **(b) Physical impairment**

- Use cooperative learning for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so they can be assisted if necessary.
- Provide these students with frequent progress checks.

## Additional tasks

### Task 6A - For slow learners

1. Find the point(s) of intersection, if any, of the line and the parabola given by  $2y + 3x - 9 = 0$  and  $y = x^2 + 2$ .

### Answer

We form the following system of equation and solve:

$$\begin{cases} y = \frac{9-3x}{2} \\ y = x^2 + 2 \end{cases}$$

$$\begin{cases} y = \frac{9-3x}{2} \\ \frac{9-3x}{2} = x^2 + 2 \end{cases}$$

$$\begin{cases} y = \frac{9-3x}{2} \\ 2x^2 + 3x - 5 = 0 \end{cases}$$

$$\begin{cases} x = \frac{-3 \pm \sqrt{9+40}}{4} \\ y = \frac{9-3x}{2} \end{cases}$$

$$\begin{cases} y = \frac{9-3x}{2} \\ x = \frac{-3 \times 7}{4} \end{cases}$$

$$\begin{cases} y = \frac{9-3(-5/2)}{2} \text{ or } y = \frac{9-3(1)}{2} \\ x = \frac{5}{2} \text{ or } x = 1 \end{cases}$$

$$\begin{cases} x = 1 \text{ or } x = \frac{5}{2} \\ y = 3 \text{ or } y = \frac{33}{4} \end{cases}$$

Thus, the solution set is  $(1, 3)$  and  $(\frac{-5}{2}, \frac{33}{4})$

### Task 6B - For talented learners

- All the students of a certain class were given a task, one sixth of all students were cleaning the dishes, thrice the square-root of all students were peeling potatoes, 12 remaining were cleaning the classroom. Find the number of all students of the class.
- A swimming pool is fitted with three pipes with uniform flow. The first two pipes operating simultaneously fill the pool in the same time during which the pool is filled by the third pipe alone. The second pipe fills the pool eight hours faster than the first pipe and one hour slower than the third pipe. Find
  - the time required by each pipe to fill the pool individually.
  - The time required by all three pipes operating simultaneously to fill the pool

### Answers

- Let  $x$  be the number of all students.

The number of students cleaning the dishes is  $\frac{x}{6}$ .

The number of students peeling potatoes is  $3\sqrt{x}$ .

The number of students cleaning the classroom is 12.

We have the equation:

$$x = \frac{x}{6} + 3\sqrt{x} + 12$$

$$x + 18\sqrt{x} - 72 = 0$$

$$5x - 18\sqrt{x} - 72 = 0$$

$$\text{Let } \sqrt{x} = y \Leftrightarrow x = y^2$$

$$5y^2 - 18y - 72 = 0$$

$$y = \frac{18 \pm \sqrt{324 + 1440}}{10} = \frac{18 \pm 42}{10}$$

$$y = 6 \text{ or } y = -\frac{24}{10} \text{ or } y = \sqrt{x} \text{ is positive}$$

Thus, the number of students in the class is 36.

- Let  $V$  be the volume of the pool. Let  $x$  be the number of hours required for the second pipe (alone) to fill the pool. The speeds (rates) at which the pool is filled by the first, second and third pipes in one hour are

$$\frac{V}{x+8}, \frac{V}{x} \text{ and } \frac{V}{x-1} \text{ respectively.}$$

Using the given information, we have

$$\frac{v}{x+8} + \frac{v}{x} = \frac{v}{x-1}$$

$$\frac{1}{x+8}, \frac{1}{x} \text{ and } \frac{1}{x-1}$$

$$\frac{x+x+8}{x(x+8)} = \frac{1}{x-1}$$

$$\frac{2x+8}{x(x+8)} = \frac{1}{x-1}$$

$$2x^2 - 2x + 8x - 8 = x^2 + 8x$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4 \text{ or}$$

$x = -2$  (this value must be rejected since we cannot have negative hours)

Thus,  $x = 4$

a) Thus, the time required by each of the three pipes to fill the pool is 12, 4, 3 hours respectively.

b) The speed of the three pipes all together to fill pool in one hour is

$$\frac{v}{12} + \frac{v}{4} + \frac{v}{3} = \frac{v+3v+4v}{12} = \frac{8v}{12} = \frac{2v}{3} = \frac{2}{3} v/h$$

The time required is  $\frac{8v}{\frac{2}{3}v/h} = \frac{v}{\frac{2v}{3h}} = \frac{v}{1} \times \frac{3h}{2v} = \frac{3}{2}h = 1h30min$

Thus, the time required by all three pipes to fill the pool is 1hour and 30 minutes.

## Assessment criteria

Model and solve algebraically or graphically daily life problems using linear equations or inequalities.

## Answers to Tasks of Unit 6 in the Student's Book

### Task 6.1 on page 99

1.  $x = \frac{1}{4}$

2. No real roots

3.  $x = 1$  or  $x = 5$                       4.  $x = -\frac{1}{2}$  or  $x = 2$   
 5.  $x = \frac{3 - \sqrt{17}}{4}$  or  $x = \frac{3 + \sqrt{17}}{4}$                       6. No real roots  
 7.  $x = -2$

**Task 6.2 on page 101**

1.  $x = -5 + 4\sqrt{2}$  or  $x = -5 - 4\sqrt{2}$                       2.  $x = 3$  or  $x = -5$   
 3.  $x = 4$  or  $x = -1$                       4.  $x = 3$  or  $x = 4$   
 5.  $x = \frac{1}{3}$  or  $x = -1$                       6.  $x = -1$  or  $x = 2$   
 7.  $x = 0$  or  $x = 2$                       8.  $x = -1$  or  $x = -\frac{1}{4}$   
 9.  $x = \frac{2}{3}$  or  $x = -1$                       10.  $x = 0$  or  $x = -\frac{1}{2}$   
 11.  $x = 0$  or  $x = -6$                       12.  $x = 0$  or  $x = 10$   
 13.  $x = 0$  or  $x = \frac{1}{2}$                       14.  $x = 5$  or  $x = -4$   
 15.  $x = 2$  or  $x = -\frac{4}{3}$                       16.  $x = 2$  or  $x = -1$   
 17.  $x = 0$  or  $x = 1$                       18.  $x = 0$  or  $x = 2$   
 19.  $x = 3$  or  $x = -1$                       20.  $x = -1$  or  $x = \frac{1}{2}$

**Task 6.3 on page 102**

- (a)  $x = -2$  or  $x = -5$                       (b)  $x = -2$  or  $x = -4$   
 (c)  $x = -1$  or  $x = -4$                       (d)  $x = 2$  or  $x = 6$   
 (e)  $x = 1$  or  $x = 4$                       (f)  $x = 3$  or  $x = 8$   
 (g)  $x = -2$  or  $x = -5$                       (h)  $x = 3$  or  $x = -5$   
 (i)  $x = 6$                       (j)  $x = 3$  or  $x = 4$   
 (k)  $x = 7$  or  $x = -6$                       (l)  $x = -2$  or  $x = -6$   
 (m)  $x = -1$  or  $x = 15$                       (n)  $x = 1$  or  $x = 2$   
 (o)  $x = 7$  or  $x = -4$                       (p)  $x = -5$  or  $x = 3$   
 (r)  $x = -5$  or  $x = 12$

**Task 6.4 on page 103**

1.  $x \in ]-\frac{1}{2}, 1[$                       2.  $x \in ]-1, \frac{3}{2}[$

3.  $x \in ]-\infty, \infty[$       4.  $x \in ]-\infty, \frac{-3-\sqrt{5}}{2} [ \cup ] \frac{-3+\sqrt{17}}{4}, +\infty[$   
 5. The answer is  $x \notin \mathbb{R}$

**Task 6.5 on page 104**

1.  $x \in ]\frac{9}{2}, -2[$       2.  $x \in ]-\infty, -2[$   
 3.  $x \in ]\frac{6}{7}, 2[$  (here the quotient  $\frac{7x-6}{2-x}$  is undefined when  $x = 2$ )

**Task 6.6 on page 105**

1.  $x \in ]-\infty, -2 [ \cup ] 1, +\infty[$       or  $x = \{-1\}$       2.  $x \in ]-\frac{1}{3}, 3[$   
 3.  $x < 11$       4.  $x \in ]-1, 0[$   
 5.  $x < 1$  and  $x > \frac{3}{2}$       6.  $x \geq \frac{5}{7}$   
 7.  $x \in ]-\infty, -1 [ \cup ] 1, 2 [ \cup ] 3, +\infty[$       8.  $x \in ]0, 1[$  and  $x > 2$   
 9.  $x < -2$  and  $x \in ]-1, 3[$       10.  $x > 4$  and  $x < -\frac{2}{3}$   
 11.  $-1 < x < 2$       12.  $x \in ]-3, -1[$  and  $x > 2$

**Task 6.7 on page 106**

1. (a)  $m < 7$       (b)  $m > 7$       (c)  $m = 7$   
 2. (a)  $k > -\frac{1}{2}$       (b)  $k < -\frac{1}{2}$       (c)  $k = -\frac{1}{2}$   
 3. (a)  $m < \frac{17}{24}$       (b)  $m > \frac{17}{24}$       (c)  $m = \frac{17}{24}$   
 4.  $]0, \frac{4}{9}[$

**Task 6.8 on page 107**

1. (a)  $x = 2, y = 4$  or  $x = -1, y = 1$       (b)  $x = 3, y = 9$  or  $x = -2, y = 4$   
 (c)  $x = 4, y = 16$  or  $x = -3, y = 9$       (d)  $x = y = 1$   
 (e)  $x = 6, y = 36$  or  $x = 8, y = 64$       (f)  $x = -4, y = 16$  or  $x = 3, y = 1$   
 (g)  $x = 1, y = 7$  or  $x = 7, y = 1$       (h)  $x = 1, y = -5$  or  $x = 5, y = -1$   
 (i)  $x = y = 2$       (j)  $x = 3, y = -2$  or  $x = -2, y = 3$   
 2. 4 and 16 or  $-2$  and 4

**Task 6.9 on page 111**

1. The building is at  $h = 64$  metres

The ball rises at the height equal to 100 metres, from the top of the building, before it starts to drop downward.

The ball hits the ground after 4 seconds, from the time of thrown.

2. George's loss is  $P(0) = -100$

The greatest possible profit is 1,006.17

3.  $t \in ] \frac{10 - \sqrt{76}}{8}, \frac{10 + \sqrt{76}}{8} [$  [or  $t \in ] 0.16, 2.34 [$

# Topic area: Analysis

## Sub-topic area: Functions

### UNIT 7

## Polynomial, rational and irrational functions

**Number of periods: 14**

### Learning objectives

By the end of this topic, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Identify a function as a rule and recognize rules that are not functions</li><li>• Determine the domain and range of a function</li><li>• Construct composition of functions</li><li>• Find whether a function is even, odd, or neither</li><li>• Demonstrate an understanding of operations on polynomials, rational and irrational functions, and find the composite of two functions.</li></ul>	<ul style="list-style-type: none"><li>• Perform operations on functions</li><li>• Apply different properties of functions to model and solve related problems in various practical contexts</li><li>• Analyse, model and solve problems involving linear or quadratic functions and interpret the results</li></ul>	<ul style="list-style-type: none"><li>• Increase self-confidence and determination to appreciate and explain the importance of functions</li><li>• Show concern on patience, mutual respect and tolerance</li><li>• When solving problems about polynomial, rational and irrational functions</li></ul>

### Content

1. Polynomials
2. Numerical functions
3. Domain and range of a function
4. Applications of rational and irrational functions

## Materials required

Pair of compasses, graph paper, rule, digital technology equipment such as calculators

## Generic competences

- Problem-solving
- Cooperation
- Communication
- Lifelong learning

## Cross-cutting issues

- **Peace and values education**

Students work harmoniously in groups.

- **Inclusive education**

Groups consist of learners of various abilities.

## Teaching and learning activities

### Introductory activity

Discuss in the groups patiently in mutual respect and tolerance, different operations on factorization.

### Main activities

1. Let learners research on the definition of a polynomial and discuss this with the rest of the class. This is Activity 7.1 on page 113 of the Student's Book.
2. Explain to them what polynomials are using examples. Give them Task 7.1 on page 114 of the Student's Book to work in pairs.
3. Let them do Activity 7.2 on page 114 of the Student's Book. Take them through the types of factorization by use of examples that include those on pages 115 to 118. Give them Task 7.2 on page 118 to attempt using the different types of factorization learnt. Allow them to explain why they choose one type over the others when solving the different problems.
4. Let them discuss and say what they know about roots of polynomials. Clarify and explain to them the concept. Help them define numerical functions by use of examples given in the Student's Book, among others.

5. Let learners explain what they understand about the domain and range of a function. Use appropriate examples to clarify the meanings and definitions. Let them attempt Task 7.3 on page 125 of the Student's Book.
6. Let students research on and discuss their findings on composition of functions. Give them Task 7.4 on page 126 of the Student's Book to tackle to ensure the concept is understood well.
7. Demonstrate what is meant by parity of functions – even functions and odd functions. Give students Task 7.5 on page 129 to attempt. Gauge their understanding of even and odd functions.
8. Let students research on applications of rational and irrational functions. This is Activity 7.3 on page 129 of the Student's Book. Let them attempt Task 7.6 on page 133.

### **Reinforcement activity**

Study algebraically polynomial functions.

### **Learners of varying strengths and abilities**

#### **(a) Gifted and talented**

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

#### **(b) Intellectual impairment**

Try to understand the specific talents of the learner and develop them. Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task. Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

## Additional tasks

### Task 7A - For slow learners

- For each of the following functions  $f$ , find its domain.
  - $f(x) = x^2 - 5$
  - $f(x) = \frac{2}{5x+6}$
- Let  $p(x) = x^3 + kx^2 + x - 6$ . Suppose that  $(x + 2)$  is a factor of  $p(x)$ .
  - Find the value of  $k$ .
  - With the value of  $k$  found in (a), factorize  $p(x)$ .

### Answers

- Since  $f(x) = x^2 - 5$  is a polynomial, then its domain is  $\text{Dom } f = \mathbb{R}$
  - $f(x) = \frac{2}{5x+6}$ . Condition:  $5x + 6 \neq 0$ .  
 $\text{Dom } f = \mathbb{R} \setminus \{-\frac{6}{5}\}$
- $p(x) = x^3 + kx^2 + x - 6$ 
  - Since  $(x - (-2))$  is a factor of  $p(x)$ , it follows from the Factor Theorem that  $p(-2) = 0$ , that is  $(-2)^3 + k(-2)^2 + (-2) - 6 = 0$ .  
Solving, we get  $k = 4$ .
  - Using long division, we get we get  $x^3 + 4x^2 + x - 6 = (x + 2)(x^2 + 2x - 3)$ .  
By inspection, we have  $p(x) = (x + 2)(x + 3)(x - 1)$ .

### Task 7B - For talented learners

- If  $\alpha$ ,  $\beta$  and  $\gamma$  are three roots of the cubic equation  $ax^3 + bx^2 + cx + d = 0$ , prove that
  - $\alpha + \beta + \gamma = -\frac{b}{a}$
  - $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$
  - $\alpha\beta\gamma = -\frac{d}{a}$

### Answers

- $ax^3 + bx^2 + cx + d = 0$   
 $a(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}) = 0$   
If  $\alpha$ ,  $\beta$  and  $\gamma$  are the roots of  $a(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}) = 0$  (1)  
then  
 $a(x - \alpha)(x - \beta)(x - \gamma) = 0$

$$a[x^3 - \beta x - \alpha x + \alpha\beta](x - \gamma) = 0$$

$$a[x^3 - \gamma x^2 - \beta x^2 + \beta\gamma x - \alpha x^2 + \alpha\gamma x + \alpha\beta x - \alpha\beta\gamma] = 0$$

$$a[x^3 + (-\alpha - \beta - \gamma)x^2 + (\alpha\beta + \beta\gamma - \alpha\gamma)x - \alpha\beta\gamma] = 0 \quad (2)$$

By equating the corresponding coefficient in (1) and (2) we get

$$\begin{cases} \frac{b}{a} = -\alpha - \beta - \gamma \\ \frac{c}{a} = \alpha\beta + \beta\gamma + \alpha\gamma \\ \frac{d}{a} = -\alpha\beta\gamma \end{cases} \quad \begin{cases} \alpha + \beta + \gamma = -\frac{b}{a} \\ \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} \text{ as required.} \\ \alpha\beta\gamma = -\frac{d}{a} \end{cases}$$

## Assessment criteria

Use concepts and definitions of functions to determine the domain of rational functions and represent them graphically.

## Answers to Tasks of Unit 7 in the Student's Book

### Task 7.1 on page 114

- Polynomial
- Polynomial
- Polynomial

### Task 7.2 on page 118

- $(a + b)(x + y)$
- $6x^2(2 + 3x)$
- $(a + b)(a + c)$
- $(px + qy)(pq + xy)$
- $(3p + 4q)^2 = (3p + 4q)(3p + 4q)$
- $(2x + 3)^2 = (2x + 3)(2x + 3)$
- $(x + 9)(x + 4)$
- $-[(2x - 3 - 3\sqrt{3})(2x - 3 + 3\sqrt{2})]$
- $-(2x - 1)(3x + 1)$
- $3(3y + 4z)(3y - 4z)$

### Task 7.3 on page 125

- Dom (f) =  $[-7, -5] \cup [3, \infty)$
  - Dom (f) =  $\mathbb{R}$
  - Dom (f) =  $\mathbb{R} \setminus \{-6\}$
  - Dom (f) =  $[2, +\infty[$

- (e)  $\text{Dom}(f) = \mathbb{R} \setminus \{-1, 2\}$   
 (f)  $\text{Dom}(f) = \mathbb{R}_+ \setminus \{\frac{7}{4}\} = [0, \frac{7}{4}[ \cup ]\frac{7}{4}, +\infty[$   
 (g)  $\text{Dom}(f) = \{x \in \mathbb{R} : x > 7\} = ]7, +\infty[$   
 (h)  $\text{Dom}(f) = ]-\infty, -1] \cup ]1, +\infty[$   
 (i)  $\text{Dom}(f) = ]-\infty, 2] \cup [3, +\infty[ = \mathbb{R} \setminus ]2, 3[$   
 (j)  $\text{Dom}(f) = \{x \in \mathbb{R} : (x \geq -\frac{2}{7}) \wedge (x \neq 7) \wedge (x \neq 1)\} = [-\frac{2}{7}, 1[ \cup ]1, 7[ \cup ]7, +\infty[$   
 (k)  $\text{Dom}(f) = \{x \in \mathbb{R} : x \neq \frac{2}{3}\} = ]-\infty, \frac{2}{3}[ \cup ]\frac{2}{3}, +\infty[$

2.  $[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}]$

3. (a)  $y \geq -3$ , (b)  $y \geq -5$  (c)  $f(x) \geq 0$ ,  
 d)  $0 \leq f(x) \leq \frac{1}{2}$   
 4. (a) 5 (b) 4 (c) 2 (d) 0

**Task 7.4 on page 126**

1. (a)  $f(g(x)) = (2x + 7)^2$  (b)  $f(g(x)) = 2x^2 + 7$   
 (c)  $f(g(x)) = \sqrt{3 - 4x}$  (d)  $f(g(x)) = 3 - 4\sqrt{x}$   
 (e)  $f(g(x)) = \frac{2}{x^2 + 3}$  (f)  $f(g(x)) = \frac{4}{x^2} + 3$   
 2. (a)  $f(x) = x^3, g(x) = 3x + 10$  (b)  $f(x) = \frac{1}{x}, g(x) = 2x + 4$   
 (c)  $f(x) = \sqrt{x}, g(x) = x^2 - 3x$  (d)  $f(x) = \frac{10}{x^3}, g(x) = 3x - x^2$

**Task 7.5 on page 129**

1. f is even 2. g is odd  
 3. h is neither even nor odd.

**Task 7.6 on page 133**

1. 10<sup>th</sup> week  
 2. 2 hours  
 3. 0.4918 FRW

# Topic area: Analysis

## Sub-topic area: Limits, differentiation and integration

### UNIT 8

## Limits of polynomial, rational and irrational functions

**Number of periods: 14**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define the concept of limit for real-valued functions of one real variable</li><li>• Evaluate the limit of a function and extend this concept to determine the asymptotes of the given function.</li></ul>	<ul style="list-style-type: none"><li>• Calculate limits of certain elementary functions</li><li>• Develop introductory calculus reasoning.</li><li>• Solve problems involving continuity.</li><li>• Apply informal methods to explore the concept of a limit including one sided limits.</li><li>• Use the concepts of limits to determine the asymptotes to the rational and polynomial functions</li></ul>	<ul style="list-style-type: none"><li>• Show concern on the importance, the use and determination of limit of functions</li><li>• Appreciate the use of intermediate-value theorem</li></ul>

### Content

1. Concept of limits:
  - Neighbourhood of a real number
  - Limit of a variable
  - One-sided limits

2. Theorems on limits:
  - Squeeze theorem
3. Indeterminate forms
4. Applications:
  - Continuity over an interval
  - Asymptotes

## Materials required

Manila papers, graph papers, ruler, markers, calculators

## Generic competences

- Communication
- Research
- Cooperation
- Problem solving
- Critical thinking

## Cross-cutting issues

- **Peace and values education**

Learners will be encouraged to avoid conflicts with their neighbours and classmates.

- **Standardization culture**

Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

Groups consist of mixed gender and all are encouraged to participate.

## Teaching and learning activities

### Introductory activity

Give them Activity 8.1 on page 136 of the Student's Book. This should enable them get the true meaning of the term neighbourhood as used in mathematics. Ensure all students participate in the discussion. Then give them Activity 8.2 on the same page to try out.

## Main activities

1. Introduce the unit by assisting learners in defining the concept of limit for real-valued functions of one real variable. Explain to learners the meaning of the limit of a variable.
2. Guide students to calculate limits of certain elementary functions. Let them attempt Task 8.1 on page 139 of the Student's Book.
3. Let them define the limit of a function and to give its graphical interpretation.
4. Guide them to discuss in groups how to evaluate the limit of a function at a point algebraically; let them extend this understanding to determine the asymptotes. Clearly explain to them the one-sided limits, the squeeze theorem, limits of functions to infinity and the various operations on limits. Use question and answer techniques to ensure full participation by learners. Let them work in pairs to tackle Task 8.2 of page 146 of the Student's Book.
5. Guide them in understanding of indeterminate cases. Take them through the various methods such as substitution and rationalisation. Let them work in pairs to tackle Activity 8.3 and Task 8.3 on page 151 of the Student's Book.
6. Let them research on the applications of limits. Guide them in a discussion and presentation of their findings in class. Ask them to do Activity 8.4 on page 153 of the Student's Book. Place emphasis on the different types of asymptotes. Let them do Activity 8.5 on page 155.
7. Let them attempt Task 8.4 on page 162 of the Student's Book.

## Reinforcement activity

Learners represent on graph papers limits of some chosen functions and draw the possible asymptotes.

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

**(b) Slow learners**

Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

**(c) Physical impairment**

- Use cooperative learning for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so as to be helped
- Provide these students with frequent progress checks.

**(d) Hearing impairment**

- Tape-record portions of textbooks, trade books, and other printed materials so students can listen (with earphones) to an oral presentation of necessary material.
- Providing written or pictorial directions to those with hearing problems
- Using of concrete objects such as models, diagrams, samples, and the like to those with impairments so as to demonstrate what you are saying by using touchable items. They can also rewrite content such as charts on large manila paper in their group work
- Facing the learner while you speak might help learners with a hearing impairment
- Use large writing on the blackboard and on visual aids

## Additional tasks

### Task 8A - For slow learners

1. Evaluate the following limits

(a)  $\lim_{x \rightarrow 2} \frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3}$

(b)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x - 2} - \sqrt{x^2 - 3x - 2})$

**Answers**

1.

$$(a) \quad \lim_{x \rightarrow 2} \frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3} = \frac{0}{0} \quad (\text{Indeterminate Form})$$

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3} &= \lim_{x \rightarrow 2} \frac{(x - \sqrt{x+2})(x + \sqrt{x+2})(\sqrt{4x+1} + 3)}{\sqrt{4x+1} - 3)(\sqrt{4x+1} + 3)(x + \sqrt{x+2})} \\ &= \lim_{x \rightarrow 2} \frac{[x^2 - (x+2)](\sqrt{4x+1} + 3)}{[(4x+1) - 9](x + \sqrt{x+2})} \\ &= \lim_{x \rightarrow 2} \frac{(x^2 - x - 2)(\sqrt{4x+1} + 3)}{(4x - 8)(x + \sqrt{x+2})} = \lim_{x \rightarrow 2} \frac{(x-2)(x+1)(\sqrt{4x+1} + 3)}{4(x-2)(x + \sqrt{x+2})} \\ &= \lim_{x \rightarrow 2} \frac{(x+1)(\sqrt{4x+1} + 3)}{4(x + \sqrt{x+2})} = \frac{(2+1)(3+3)}{4(2+2)} = \frac{3(6)}{4(4)} = \frac{18}{16} = \frac{9}{8} \end{aligned}$$

$$(b) \quad \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 5x - 2} - \sqrt{x^2 - 3x - 2}) = \infty - \infty \quad (\text{Indeterminate form})$$

$$\lim_{x \rightarrow +\infty} = \frac{(\sqrt{x^2 + 5x - 2} - \sqrt{x^2 - 3x - 2})(\sqrt{x^2 + 5x - 2} + \sqrt{x^2 - 3x - 2})}{(\sqrt{x^2 + 5x - 2} + \sqrt{x^2 - 3x - 2})}$$

$$\lim_{x \rightarrow +\infty} = \frac{(x^2 + 5x - 2 - x^2 + 3x + 2)}{(\sqrt{x^2 + 5x - 2} + \sqrt{x^2 - 3x - 2})}$$

$$\lim_{x \rightarrow +\infty} = \frac{8x}{\sqrt{x^2(1 + \frac{5}{x} - \frac{2}{x^2})} + \sqrt{x^2(1 - \frac{3}{x} - \frac{2}{x^2})}}$$

$$\begin{aligned} &= \lim_{x \rightarrow +\infty} = \frac{8x}{x \left( \sqrt{1 + \frac{5}{x} - \frac{2}{x^2}} + \sqrt{1 - \frac{3}{x} - \frac{2}{x^2}} \right)} = \lim_{x \rightarrow +\infty} \frac{8}{2} = 4 \end{aligned}$$

### Task 8B - For talented learners

1. Given the function  $f(x) = \frac{ax^2}{ax^2 + 6x + c}$  it is given that its curve representative has three asymptotes of respective equations  $x - 2 = 0$ ,  $x - 1 = 0$ ,  $y + 4 = 0$ . Find the real values of  $a$ ,  $b$  and  $c$ .

#### Answer

1.  $f(x) = \frac{ax^2}{ax^2 + 6x + c}$   
 $x = 2$  and  $x = 1$  are vertical asymptotes, thus 2 and 1 are the roots of the equation  $bx^2 + 6x + c = 0$ .

$$\begin{cases} 4b + 12 + c = 0 \\ b + 6 + c = 0 \end{cases}$$

$$\begin{cases} 4b + c = -12 \\ b + c = -6 \end{cases}$$

$$\begin{cases} 3b = -6 \\ b + c = -6 \end{cases}$$

$$\begin{cases} b = -2 \\ c = -6 - b \end{cases}$$

$$\begin{cases} b = -2 \\ c = -4 \end{cases}$$

$y = -4$  is an horizontal asymptote

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{ax^2}{ax^2 + 6x + c} = \frac{a}{b}. \text{ So } \frac{a}{b} = -4 \Rightarrow \frac{a}{-2} = -4 \Rightarrow a = 8$$

$$\begin{cases} a = 8 \\ b = -2 \\ c = -4 \end{cases}$$

## Answers to Tasks of Unit 8 in the Student's Book

### Task 8.1 on page 139

1.  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$

2.  $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$

3.  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

4.  $\lim_{x \rightarrow 1^-} f(x) = -1$ ,  $\lim_{x \rightarrow 1^+} f(x) = 1$  and

$\lim_{x \rightarrow 1} f(x)$  does not exist

### Task 8.2 on page 146

1. (a)  $\lim_{x \rightarrow 2} 2x + 1 = 2(2) + 1 = 5$

(b)  $\lim_{a \rightarrow 1} a^2 - 1 = 1 - 1 = 0$

(c)  $\lim_{x \rightarrow -3} \frac{x^2 + x - 2}{x + 1} = \frac{9 - 3 - 2}{-3 + 1} = \frac{4}{-2} = -2$

$$\begin{aligned}
 \text{(d)} \quad \lim_{x \rightarrow -\infty} \frac{4x^2 + 3x - 2}{x^2 + x - 1} &= \lim_{x \rightarrow -\infty} \frac{x^2(4 + \frac{3}{x} - \frac{2}{x^2})}{x^2(1 + \frac{1}{x} - \frac{1}{x^2})} = \frac{\lim_{x \rightarrow -\infty} (4 + \frac{3}{x} - \frac{2}{x^2})}{\lim_{x \rightarrow -\infty} (1 + \frac{1}{x} - \frac{1}{x^2})} \\
 &= \frac{\lim_{x \rightarrow -\infty} 4 + \lim_{x \rightarrow -\infty} \frac{3}{x} - \lim_{x \rightarrow -\infty} \frac{2}{x^2}}{\lim_{x \rightarrow -\infty} 1 + \lim_{x \rightarrow -\infty} \frac{1}{x} - \lim_{x \rightarrow -\infty} \frac{1}{x^2}} = \frac{4 + 0 - 0}{1 + 0 - 0} = \frac{4}{1} = 4
 \end{aligned}$$

2. (a) We know that  $-1 \leq \cos 2x \leq 1$

$$-x \leq x \cos 2x \leq x$$

$$\lim_{x \rightarrow 0} -x = 0 = \lim_{x \rightarrow 0} x$$

Therefore,  $\lim_{x \rightarrow 0} x \cos 2x = 0$

(b) We know that  $-1 \leq \sin x \leq 1$

Since  $x \in [0, +\infty[$ , we have

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

We know that  $\lim_{x \rightarrow +\infty} -\frac{1}{x} = 0 = \lim_{x \rightarrow +\infty} \frac{1}{x}$

Therefore,  $\lim_{x \rightarrow +\infty} \frac{\sin x}{x} = 0$

(c) We know that if  $0 \leq x \leq \frac{\pi}{2}$  then  $\sin x \leq x \leq \tan x$

Dividing each member of this inequality by  $\sin x$  gives

$$\frac{\sin x}{\sin x} \leq \frac{x}{\sin x} \leq \frac{\tan x}{\sin x}$$

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Inverting member by member gives

$$1 \geq \frac{\sin x}{x} \geq \cos x \Leftrightarrow \cos x \leq \frac{\sin x}{x} \leq 1$$

Since  $\lim_{x \rightarrow 0} \cos x = 1 = \lim_{x \rightarrow 0} 1$

Therefore  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

### Task 8.3 on page 151

- |                       |                            |                          |                            |
|-----------------------|----------------------------|--------------------------|----------------------------|
| 1. 3                  | 2. $-\frac{3}{4}$          | 3. 4                     | 4. $\frac{1}{2\sqrt{3}}$   |
| 5. -1                 | 6. $\frac{1}{2}$           | 7. $\frac{1}{2}$         | 8. $-\frac{2}{3}$          |
| 9. -2                 | 10. $-\frac{1}{2\sqrt{3}}$ | 11. $-\frac{5}{2}$       | 12. $-\frac{3}{4}$         |
| 13. $-\frac{3}{4}$    | 14. $\frac{1}{2\sqrt{5}}$  | 15. 9                    | 16. 7                      |
| 17. $-\frac{1}{3}$    | 18. 0                      | 19. $\frac{1}{5}$        | 20. $-\infty$              |
| 21. -1                | 22. 1                      | 23. -9                   | 24. 5                      |
| 25. 0                 | 26. 7                      | 27. 3                    | 28. 0                      |
| 29. 1                 | 30. $-\infty$              | 31. 625                  | 32. $-\frac{1}{6}$         |
| 33. 4                 | 34. 3                      | 35. $-\infty$            | 36. 0                      |
| 37. $\frac{9}{8}$     | 38. 0                      | 39. Does not exist       | 40. $-\frac{4}{3}$         |
| 41. 3                 | 42. $\frac{13}{3}$         | 43. $\frac{1}{\sqrt{2}}$ | 44. $\frac{3}{7}$          |
| 45. -1                | 46. $\frac{3}{4}$          | 47. $\frac{1}{4}$        | 48. $\infty$               |
| 49. $\frac{1}{4}$     | 50. -3                     | 51. $\frac{1}{3}$        | 52. $\frac{1}{6}$          |
| 53. -15               | 54. 0                      | 55. $\frac{1}{6}$        | 56. $-\frac{3}{2}$         |
| 57. 1                 | 58. $\frac{1}{4}$          | 59. $\frac{7}{4}$        | 60. $\frac{1}{12}$         |
| 61. (a) $\frac{1}{2}$ | (b) $\frac{1}{4}$          | (c) $\frac{2}{3}$        | (d) $\frac{2-\sqrt{2}}{2}$ |
| (e) 1                 |                            |                          |                            |

$$62. |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \text{ therefore } \frac{|x| - x}{x} = \begin{cases} \frac{x-x}{x} = 0 \\ \frac{-x-x}{x} = -2 \end{cases}$$

$$(a) \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} \frac{|x| - x}{x} = \lim_{x \rightarrow 0^+} 0 = 0$$

$$(b) \lim_{x \rightarrow 0^-} y = \lim_{x \rightarrow 0^+} \frac{|x| - x}{x} = \lim_{x \rightarrow 0^+} -2 = -2$$

$$(c) \text{ Since } \lim_{x \rightarrow 0^+} y \neq \lim_{x \rightarrow 0^-} y \text{ then } \lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} \frac{|x| - x}{x} \text{ does not exist.}$$

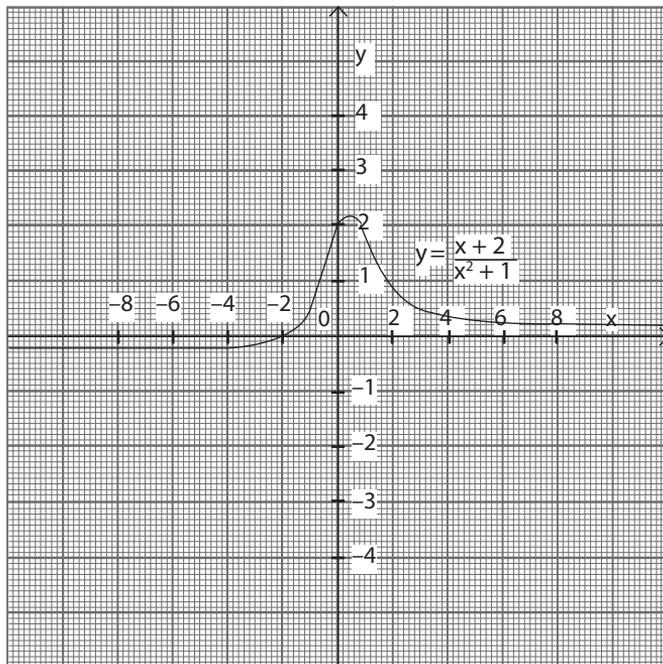
$$(d) \lim_{x \rightarrow 0^-} y = \lim_{x \rightarrow 0^+} \frac{|x| - x}{x} = \lim_{x \rightarrow 0^+} -2 = -2$$

### Task 8.4 on page 162

1. Horizontal Asymptote H.A  $\equiv y = 0$ .

No Vertical Asymptote. No Oblique Asymptote

Graph

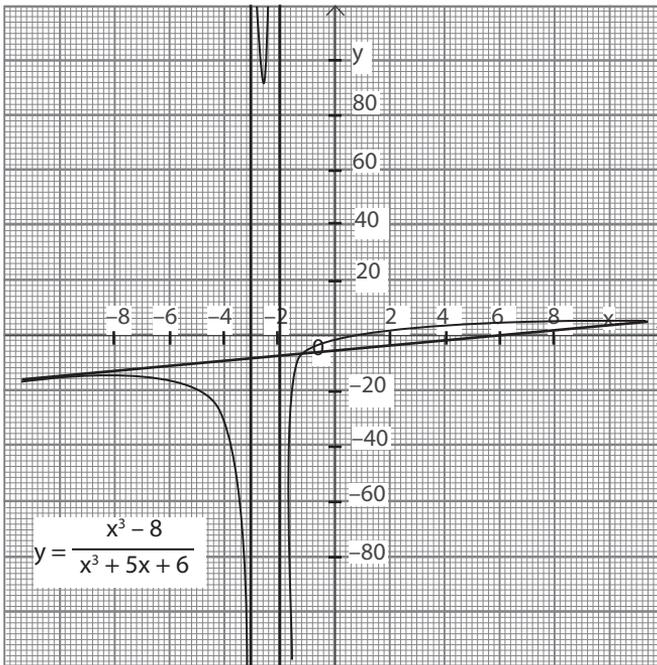


2. Vertical Asymptote V.A  $\equiv x = -3$  Vertical Asymptote V.A  $\equiv x = -2$

Oblique Asymptote O.A  $\equiv y = x - 5$

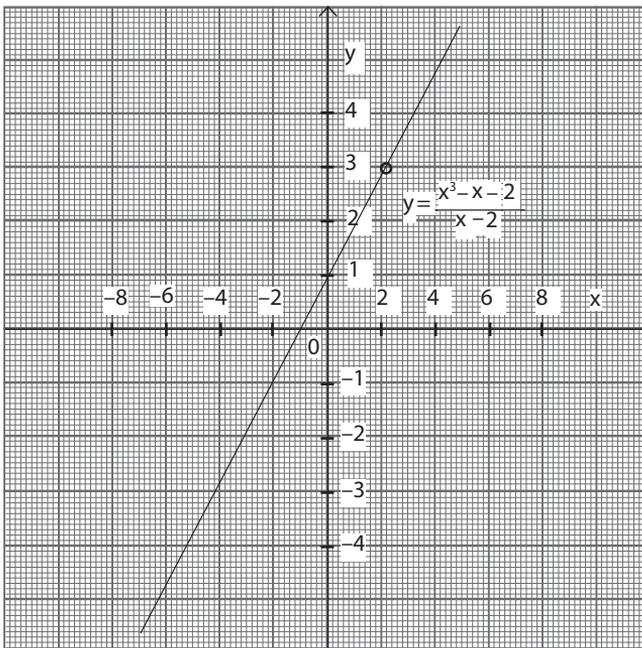
No horizontal Asymptote

## Graph



3. This can be simplified to a linear function  $x+1$ . It has a whole (not defined) at  $x = 2$ . It does not have any asymptote to it.

## Graph



# Topic area: Analysis

Sub-topic area: Limits, differentiation and integration

UNIT  
**9**

## Differentiation of polynomials, rational and irrational functions and their application

**Number of periods: 21**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Evaluate derivatives of functions using the definition of derivative</li><li>• Define and evaluate from first principles the gradient at a point</li><li>• Distinguish between techniques of differentiation to use in an appropriate context</li></ul>	<ul style="list-style-type: none"><li>• Use properties of derivatives to differentiate polynomial, rational and irrational functions</li><li>• Use first principles to determine the gradient of the tangent line to a curve at a point</li><li>• Apply the concepts of and techniques of differentiation to model, analyse and solve rates or optimisation problems in different situations</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the use of gradient as a measure of rate of change (economics)</li><li>• Appreciate the importance and use of differentiation in Kinematics (velocity, acceleration )</li><li>• Show concern on derivatives to help in the understanding optimization problems</li></ul>

## Content

1. Concepts of derivatives of a function:
  - Definition
  - Differentiation from first principles
  - High order derivatives
2. Rules of differentiation
3. Application of differentiation:
  - Geometric interpretation of derivatives
  - Mean value theorem for derivatives
  - Variations of a function
  - Rates of change problems
  - Optimization problems...

## Materials required

Manila papers, graph papers, Digital technology including calculators, ...

## Generic competences

- Communication
- Problem-solving
- Research
- Cooperation
- Critical thinking

## Cross-cutting issues

- **Comprehensive sexuality education**  
Learners taught to work together with opposite gender without the need to have ulterior motives.
- **Standardization culture**  
Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.
- **Gender education**  
Groups consist of mixed gender and all are encouraged to participate.

## Teaching and learning activities

### Introductory activity

The students have encountered gradients before. Ask them what they recall about the concept of gradients. Let them do the Mental task on page 165 of the Student's Book.

### Main activities

1. Introduce the topic by defining differentiation. Let students do Activity 9.1 on page 166 of the Student's Book. Then guide learners to define and evaluate, from first principles, the gradient at a point. Give them Activity 9.2 on page 169 of the Student's Book to try out in pairs. Let them individually tackle Task 9.1 on page 170.
2. Demonstrate to students how to use properties of derivatives to differentiate polynomial, rational and irrational functions. Let them attempt Task 9.2 on page 172.
3. Introduce them to higher order derivatives. Then outline the rules of differentiation. Let them tackle Tasks 9.3, 9.4 and 9.5 on pages 174, 178 and 180 respectively, one at a time, after every subtopic.
4. Guide the learners on how they can distinguish between the different techniques of differentiation and to use them in an appropriate context. Let them do Task 9.6 on page 182 and Task 9.7 on page 185 to reinforce concepts involving theorems. Let them do Activity 9.3 on page 187 and Task 9.8 on page 189.
5. Introduce them to the concept of increasing and decreasing functions. They should be able to explain the sign of a derivative, stationary and inflection points, and concavity. Give them Task 9.9 on page 195.
6. Help the students to use the different techniques of differentiation to model, analyse and solve rates and optimization problems. Follow up with the mental task on page 195, Task 9.10 on page 196, Task 9.11 on page 198 and Activity 9.4 on page 198. Give them Task 9.12 on page 200 of the Student's Book.

### Reinforcement activities

1. Represent on graph papers the gradient of a straight line and interpret it geometrically.
2. Determine the gradient of different functions at a point using the definition of derivatives, from the first principles and chain rule, and interpret the results.

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

### (b) Intellectual impairment

- Try to understand the specific talents of the learner and develop them.
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

### Task 9A - For slow learners

1. Find the derivative of the following function using the definition of the derivative:  $g(t) = \sqrt{t}$

#### Answer

$$\begin{aligned} 1. \quad \text{Find } g(t) = \sqrt{t} \text{ then } \frac{dg}{dt} &= \lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} = \frac{0}{0} \text{ (Indet. Form)} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{t+h} - \sqrt{t})(\sqrt{t+h} + \sqrt{t})}{h(\sqrt{t+h} + \sqrt{t})} = \lim_{h \rightarrow 0} \frac{t+h-t}{h(\sqrt{t+h} + \sqrt{t})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{t+h} + \sqrt{t})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{\sqrt{t+0} + \sqrt{t}} = \frac{1}{2\sqrt{t}} \end{aligned}$$

### Task 9B - For talented learners

1. A rectangular field of 200 m of perimeter is to be enclosed. If we want a maximum of area, find the measures of its sides and the corresponding area.
2. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

## Answers

1. Let  $x$  and  $y$  be the width and the length of the field respectively.

$$2(x + y) = 200$$

$$x + y = 100$$

$$y = 100 - x$$

$$\text{The area is } xy = x(100 - x) = 100x - x^2$$

Let  $f(x) = 100x - x^2$ . to find the largest value of area, is like to find the maximum of the function,  $f(x)$ .

$$f'(x) = 100 - 2x$$

$$f'(x) = 0 \quad 100 - 2x = 0$$

$$2x = 100$$

$$x = 50$$

If  $x = 50$  then  $y = 50$ . That gives us the area of  $50 \times 50 = 2500$ .

The rectangular field is a square of side 50m and thus its area is  $2500\text{m}^2$ .

2. Let the radius of the base of the cylinder be  $r$  and height be  $h$ . Let  $S$ : surface,  $V$ : volume.

$$S = 2\pi rh + 2\pi r^2 \text{ or } h = \frac{S - 2\pi r^2}{2\pi r}$$

$$V = \pi r^2 h = \pi r^2 \left( \frac{S - 2\pi r^2}{2\pi r} \right)$$

$$V = \frac{1}{2}(Sr - 2\pi r^3)$$

$$\frac{dV}{dr} = \frac{1}{2}(S - 6\pi r^2)$$

For maximum or minimum volume  $\frac{dV}{dr} = 0$

$$\frac{1}{2}(S - 6\pi r^2) = 0 \quad S = 6\pi r^2$$

$$h = \frac{S - 2\pi r^2}{2\pi r} = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r$$

Therefore, the cylinder has maximum volume if the height is equal to the diameter.

## Additional information

### ***Proof of the rules of differentiation***

Each of the rules can be proved using the first principles definition of .

The following proof are worth examining.

1. If  $f(x) = cu(x)$  where  $c$  is a constant then  $f'(x) = cu'(x)$  .

**Proof:**

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c u(x+h) - c u(x)}{h} = \lim_{h \rightarrow 0} c \left[ \frac{u(x+h) - u(x)}{h} \right] \\ &= c \lim_{h \rightarrow 0} \left[ \frac{u(x+h) - u(x)}{h} \right] = c u'(x) \end{aligned}$$

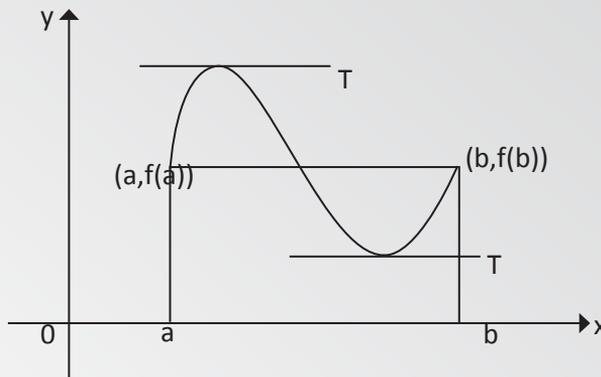
2. If  $f(x) = u(x) + v(x)$  then  $f'(x) = u'(x) + v'(x)$

**Proof:**

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \left( \frac{u(x+h) + v(x+h) - [u(x) + v(x)]}{h} \right) \\ &= \lim_{h \rightarrow 0} \left( \frac{u(x+h) - u(x) + v(x+h) - v(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} = u'(x) + v'(x) \end{aligned}$$

### **Geometrical significance of Rolle's Theorem**

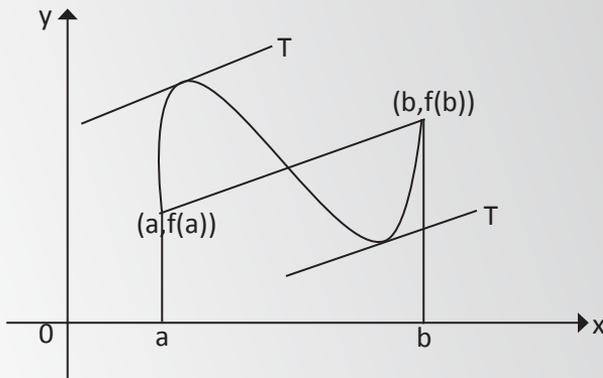
Rolle's Theorem states that the curve representing of graph of the function  $f$  must have a tangent parallel to  $x$ -axis, at least at one point between and.



### **Mean value theorem**

The expression  $\frac{f(b) - f(a)}{b - a}$  represent the slope of the chord joining the points  $(a, f(a))$  and  $(b, f(b))$ .

Then, since  $f'(x)$  is the slope of the tangent line to the curve, the mean theorem states that there is always a number  $c$  between  $a$  and  $b$  such that the slope of the tangent at the point  $(c, f(c))$  is the same as the slope of the chord. i.e tangent line and chord are parallel.



## Assessment criteria

Use the gradient of a straight line as a measure of rate of change and apply this to line tangent or normal curves in various contexts. Use these concepts to solve and interpret related rates and optimization problems in various contexts.

## Answers to Tasks of Unit 9 in the Student's Book

### Task 9.1 on page 170

- (a) 0                      (b)  $1 + 3x^2$             (c)  $3x^2 + 2$             (d)  $4x^3 - \frac{1}{3}$
- (a)  $4 + 4x$               (b) 12                      (c) -4
- (a)  $-\frac{1}{(x+2)^2}$           (b)  $-\frac{2}{(2x-1)^2}$           (c)  $-\frac{2}{x^3}$                 (d)  $-\frac{3}{x^3}$
- (a)  $\frac{1}{2\sqrt{x+2}}$           (b)  $-\frac{1}{2x\sqrt{x}}$             (c)  $\frac{1}{2\sqrt{2x+2}}$

### Task 9.2 on page 172

- (a) 4                      (b) -4                      (c) 5                      (d) 1  
 (e) -12                  (f) 3
- (a) -1                    (b) -1                    (c)  $-\frac{3}{4}$                     (d) 1  
 (e)  $-\frac{1}{32}$                   (f) -12                    (g) -9                    (h)  $-\frac{45}{289}$
- (a)  $\frac{1}{2}$                     (b) 1                      (c)  $-\frac{1}{27}$                     (d)  $\frac{1}{4}$

### Task 9.3 on page 174

- $f'(x) = 3x^2 + 2$  and  $f''(x) = 6x$
- $f'(x) = -\frac{7}{(x-2)^2}$  and  $f''(x) = \frac{14}{(x-2)^3}$

- (c)  $f'(x) = -\frac{2}{x^2}$  and  $f''(x) = \frac{4}{x^3}$
- (d)  $f'(x) = \frac{x^2 + 4x + 1}{(x+2)^2}$  and  $f''(x) = \frac{6}{(x+2)^3}$
- (e)  $f'(x) = 4x - \frac{1}{x^2} - \frac{1}{2\sqrt{x}}$  and  $f''(x) = 4 + \frac{2}{x^3} + \frac{1}{4x\sqrt{x}}$
- (f)  $f'(x) = \frac{5x^2 + 2x + 7}{(x^2 + 2x - 1)^2}$  and  $f''(x) = \frac{-10x^3 - 6x^2 - 42x - 30}{(x^2 + 2x - 1)^3}$

### Task 9.4 on page 178

1. (a)  $4x$  (b)  $2x + 3$  (c)  $3x^2 + 6x + 4$
- (d)  $20x^3 - 12x$  (e)  $\frac{6}{x^2}$  (f)  $-\frac{2}{x^2} + \frac{6}{x^3}$
- (j)  $2x - \frac{5}{x^2}$  (h)  $2x + \frac{3}{x^2}$  (i)  $-\frac{1}{2x\sqrt{x}}$
- (j)  $8x - 4$  (k)  $3x^2 + 12x + 12$
2. a)  $3x^2 + 6x$  (b)  $10x$  (c)  $-\frac{2}{5x^3}$
- d)  $100$  (e)  $15$  (f)  $12x^2$
3. a)  $1$  (b)  $\frac{3\sqrt{x}}{2}$  (c)  $2x - 10$
- d)  $2 - 9x^2$  (e)  $4 + \frac{1}{4x^2}$  (f)  $-11$
4. (a)  $2$  (b)  $-\frac{16}{729}$  (c)  $-7$
- (d)  $\frac{13}{4}$  (e)  $\frac{1}{8}$  (f)  $-11$
5. (a)  $2 - \frac{1}{2\sqrt{x}}$  (b)  $\frac{1}{3\sqrt[3]{x^2}}$  (c)  $\frac{1}{x\sqrt{x}}$
- (d)  $\frac{2}{\sqrt{x}} + 1$  (e)  $-\frac{2}{x\sqrt{x}}$  (f)  $6x - \frac{3}{2}\sqrt{x}$
- (g)  $-\frac{25}{2x^3\sqrt{x}}$  (h)  $2 + \frac{9}{2x^2\sqrt{x}}$
6. (a)  $\frac{dy}{dx} = 3 + \frac{2}{x^2}$ ,  $\frac{dy}{dx}$  is the slope function of  $y = 3x - \frac{2}{x}$  from which the slope at any point can be found.
- (b)  $\frac{ds}{dt} = 2t + 6$  metres per second,  $\frac{ds}{dt}$  is the instantaneous rate of change in position at the time  $t$ , i.e.. it is the velocity function.
- (c)  $\frac{dC}{dx} = 4 + \frac{1}{200}x$  FRW per pen,  $\frac{dC}{dx}$  is the instantaneous rate of change in cost as the number of pens changes.

**Task 9.5 on page 180**

1. (a)  $8(4x - 5)$  (b)  $2(5 - 2x)^{-2}$   
 (c)  $\frac{1}{2}(3x - x^2)^{-1/2}(3 - 2x)$  (d)  $-12(1 - 3x)^2$   
 (e)  $-18(5 - x)^2$  (f)  $\frac{1}{3}(2x^3 - x^2)^{-2/3}(6x^2 - 2x)$   
 (g)  $-60(5x - 4)^{-3}$  (h)  $-4(3x - x^2)^{-2}(3 - 2x)$   
 (i)  $6(x^2 - \frac{2}{x})^2(2x + \frac{2}{x^2})$
2. (a)  $-\frac{1}{\sqrt{3}}$  (b)  $-18$  (c)  $-8$   
 (d)  $-6$  (e)  $-\frac{3}{32}$  (f)  $0$
3. (a)  $\frac{dy}{dx} = 3x^2,$   
 (b)  $\frac{dx}{dy} = \frac{1}{3}y^{-2/3}$  .....here we substitute  $y = x^3$

**Task 9.6 on page 182**

1. (a)  $2x(2x - 1) + 2x^2$  (b)  $2x(x - 3)(2x - 3)$   
 (c)  $2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}}$  (d)  $\frac{3(x-1)}{2\sqrt{x}}$   
 (e)  $\frac{x^2(7x-6)}{2\sqrt{x-1}}$  (f)  $\frac{1}{2\sqrt{x}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x)$
2. (a)  $-48$  (b)  $-3$   
 (c)  $\frac{13}{3}$  (d)  $\frac{11}{2}$
3.  $x = 3$  or  $\frac{3}{5}$
4. (a)  $\frac{2}{(x+2)^2}$  (b)  $\frac{9}{(x+5)^2}$   
 (c)  $\frac{(x^2-3)-2x^2}{(x^2-3)^2}$  (d)  $\frac{6-8x}{(2x+3)^4}$   
 (e)  $\frac{2x(3x-x^2)-(x^2-3)(3-2x)}{(3x-x^2)^2}$  (f)  $\frac{\sqrt{1-3x} + \frac{3x}{2\sqrt{1-3x}}}{1-3x}$   
 (g)  $\frac{2}{(x^2+2)^{3/2}}$
5. (a)  $1$  (b)  $1$  (c)  $-\frac{7}{324}$   
 (d)  $-\frac{28}{27}$

**Task 9.7 on page 185**

1. (a)  $T \equiv y = 2x - 5$ ,  $N \equiv 2y + x + 5 = 0$   
(b)  $T \equiv y = 4x - 2$ ,  $N \equiv 4y + x + 8 = 0$   
(c)  $T \equiv y + x + 2$ ,  $N \equiv y = x$   
(d)  $T \equiv y = 5$ ,  $N \equiv x = 0$   
(e)  $T \equiv y + x = 3$ ,  $N \equiv y = x - 1$   
(f)  $T y = 19x + 26$ ,  $N \equiv 19y + x + 230 = 0$
2. (a)  $y = -7x + 11$  (b)  $4y = x + 8$   
(c)  $y = -2x - 2$  (d)  $y = -2x + 6$
3. (a)  $6y = -x + 57$  (b)  $7y = -x + 36$   
(c)  $3y = x + 11$  (d)  $x + 6y = 43$
4. (a)  $(-\frac{1}{2}, 2\sqrt{2})$  (b)  $y = 21$  and  $y = -6$   
(c)  $m = -5$  (d)  $y = -3x + 1$
5. (a)  $3y = x + 5$  (b)  $9y = x + 4$   
(c)  $16y = x - 3$  (d)  $y = -4$
6. (a)  $y = 2x - \frac{7}{4}$  (b)  $y = -27x - \frac{242}{3}$   
(c)  $57y = -4x + 1042$  (d)  $2y = x + 1$
7.  $m = 4$ ,  $n = 3$

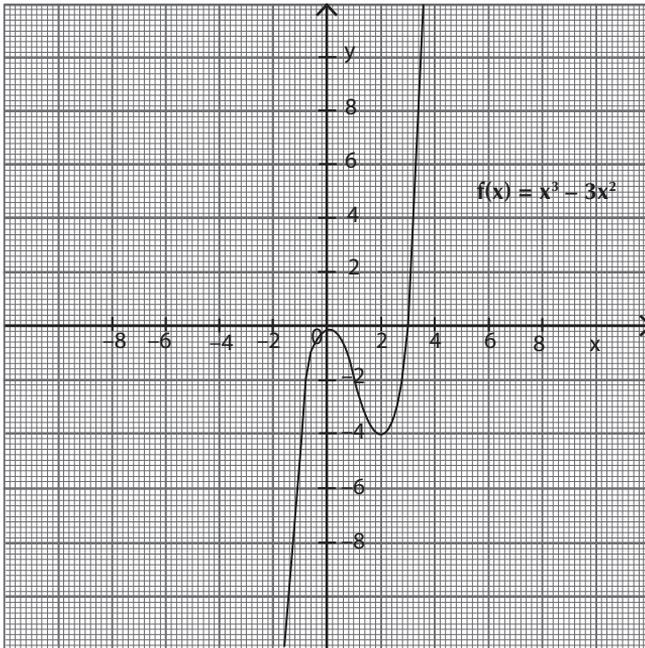
**Task 9.8 on page 189**

1. 0                      3. 3                      4.  $-\frac{7}{3}$                       5. -4
2. -1

**Task 9.9 on page 195**

- a) i.  $]-\infty, 0 [ \cup ] 2, +\infty [$   
ii)  $x = 0$  or  $x = 2$   
iii)  $] 1, +\infty [$   
iv)  $x = 1$   
v) The extreme points are  $(0, 0)$  and  $2, -4$ . The point of inflection is  $(1, -2)$

## Graph



b)

i)  $]-\infty, 0 [ \cup ] 3, +\infty [$

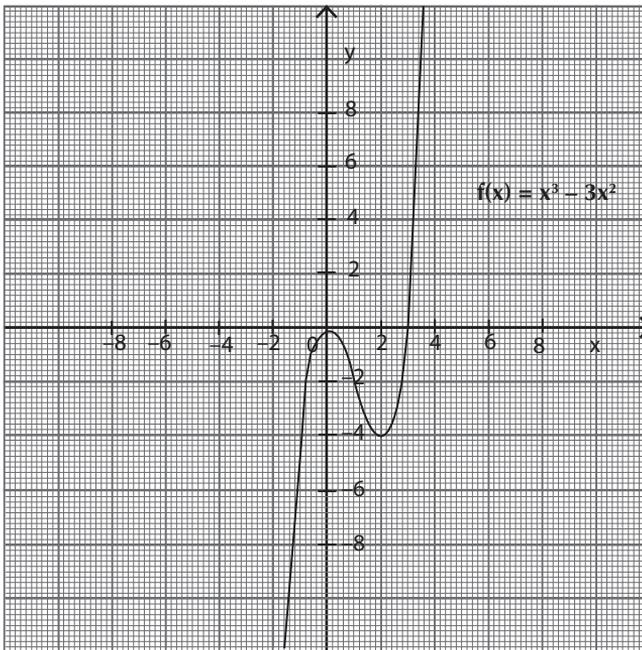
ii)  $x = 0$  or  $x = 3$

iii)  $]-\infty, 0 [ \cup ] 2, +\infty [$

iv)  $x = 0$  or  $x = 2$

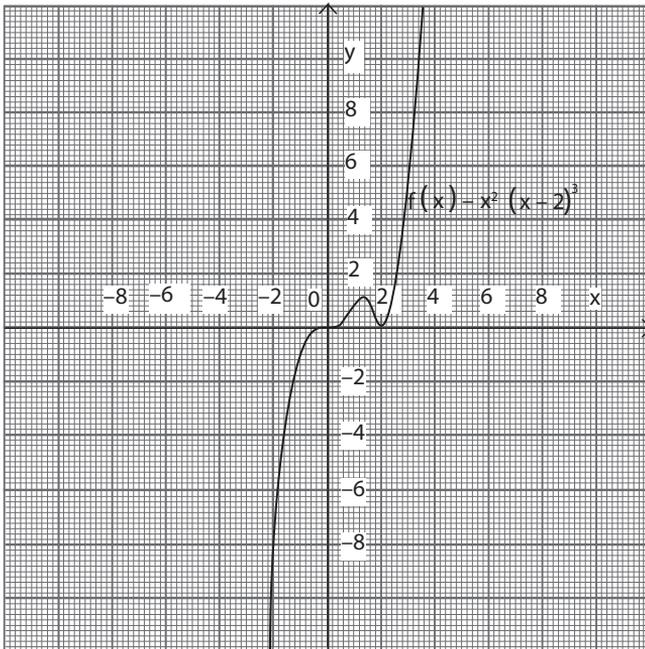
v) The extreme points are (0, 0) and (3, -27). The points of inflection are (0, 0) and (2, -16).

## Graph



- c) (i)  $] -\infty, \frac{6}{5} [ \cup ] 2, +\infty [$
- (ii)  $x = 0$  or  $x = 2$  or  $x = \frac{6}{5}$
- (iii)  $] 0, \frac{6 - \sqrt{6}}{5} [ \cup ] \frac{6 + \sqrt{6}}{5}, +\infty [$
- (iv)  $x = 0$  or  $x = \frac{6 - \sqrt{6}}{5}$  or  $x = \frac{6 + \sqrt{6}}{5}$
- (v) Extreme points are  $(0, 0)$ ,  $(2, 0)$  and  $(\frac{6}{5}, \frac{3456}{3125})$ . The inflection points are at  $(0, 0)$ ,  $(\frac{6 - \sqrt{6}}{5}, f(\frac{6 - \sqrt{6}}{5}))$  and  $(\frac{6 + \sqrt{6}}{5}, f(\frac{6 + \sqrt{6}}{5}))$

## Graph



### Task 9.10 on page 196

- Let  $x$  be the side of a square sheet. If  $A$  is the area of a square sheet at time  $t$ , then we have  $A = x^2$ . Differentiating both sides with respect to time  $t$ , we get  $\frac{dA}{dt} = 2x \frac{dx}{dt}$ . Rate of increase of side  $x$  with respect to time  $t$  is 4 cm/sec. i.e.  $\frac{dx}{dt} = 4$  cm/sec.

Then, the rate of change of area  $A$  with respect to time  $t$  is  $\frac{dA}{dt} = 2x \times 4 = 8x$ .  
Hence, rate of increase of area  $A$  with respect to  $t$  when  $x = 8$  is  $8 \times 8 = 64$  cm<sup>2</sup>/sec.

- Let the radius of circle be  $r$  (in cm)

Given: Rate of change of radius be  $\frac{dr}{dt} = 3$  cm.

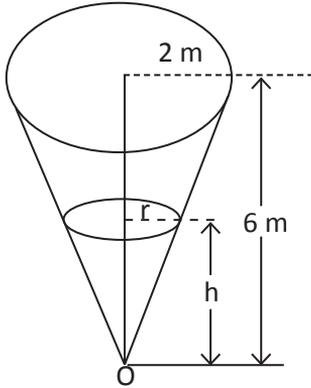
Area of circle  $A = \pi r^2$ .

Differentiating both sides with respect to time  $t$ , we get  $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

When  $\frac{dr}{dt} = 3$  therefore  $\frac{dA}{dt} = 2\pi r \times 3 = 6\pi r$

When  $r = 10$  the rate of change of area is  $\frac{dA}{dt} = 6\pi \times 10 = 60$  cm<sup>2</sup>/sec .

- Let  $r$  be the radius of the cone of oil where  $h$  denotes the height of the oil at a given time. Now  $V = \frac{\pi r^2 h}{3}$ .



From the figure, using similar triangles.

$$\frac{r}{R} = \frac{h}{H} \Rightarrow \frac{r}{2} = \frac{h}{6} \Rightarrow r = \frac{h}{3}$$

Then, the volume of oil at height  $h$  is given by  $V = \frac{\pi(\frac{h}{3})^2 h}{3} = \frac{\pi h^3}{27}$

Differentiating both sides with respect to time  $t$ , we get

$$\frac{dv}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$$

Given that  $\frac{dv}{dt} = 0.002 \text{ m}^3/\text{min}$

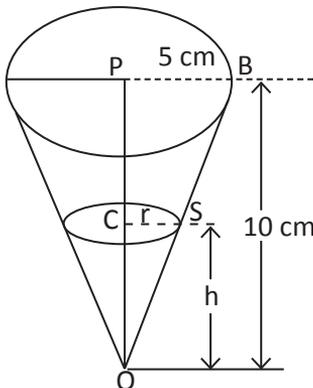
For  $h = 3 \text{ m}$ , we get

$$0.002 = \frac{\pi}{9} (3)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{0.002}{3.14} = 6.37 \times 10^{-4} \text{ m/min}$$

Hence, the rate at which the height of the oil is increasing is  $6.37 \times 10^{-4} \text{ m/min}$ .

4. Let  $r$  (in cm) be the radius of the circular water-surface when the depth is  $h$  cm; at the end of time  $t$  minutes.



From figure, using similar triangles;

$$\frac{r}{R} = \frac{h}{H} \Rightarrow \frac{r}{5} = \frac{h}{10} \Rightarrow r = \frac{h}{2}$$

Let  $v$  be the volume of water at time  $t$  minutes.

$$\text{Then } v = \frac{\pi(\frac{h}{2})^2 h}{3} = \frac{\pi h^3}{12}$$

$$\text{Now } v = \frac{\pi h^3}{12}$$

Differentiating both sides with respect to t, we get

$$\frac{dv}{dt} = \frac{3}{12} \pi h^2 \frac{dh}{dt} = \frac{1}{4} \pi h^2 \frac{3}{8\pi}$$

Given that  $\frac{dv}{dt} = 1.5 \text{ cm}^3/\text{min}$  and  $h = 4 \text{ cm}$ , we get

$$\frac{1}{4} \pi (4)^2 \frac{dh}{dt} = 1.5$$

$$4\pi \frac{dh}{dt} = 1.5$$

$$\frac{dv}{dt} = \frac{1.5}{4\pi} = \frac{15}{40\pi} = \frac{3}{8\pi} \text{ cm/min}$$

Hence, the rate at which the level of water is rising is  $\frac{3}{8\pi} \text{ cm/min}$ .

5. Suppose that each tin has radius  $r$  and perpendicular height  $h$  and  $S$  is the total area. We require the dimensions for the surface area  $S$  to be minimum and so we need an expression for  $S$  which is  $S = 2\pi r^2 + 2\pi rh$ . We cannot differentiate this expression in this form as  $S$  is given as a function of two variables  $r$  and  $h$ .

However the volume is  $V = \pi r^2 h$  and this volume is to be  $128\pi \text{ cm}^3$ .

$$128\pi = \pi r^2 h \Leftrightarrow h = \frac{128}{r^2} \quad (*)$$

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \left( \frac{128}{r^2} \right) = 2\pi r^2 + \frac{256\pi}{r} \text{ and } \frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$$

$$\text{When } \frac{dS}{dr} = 0 \text{ then } 4\pi r - \frac{256\pi}{r^2} = 0 \Leftrightarrow \frac{4\pi r^3 - 256\pi}{r^2} = 0 \Leftrightarrow 4\pi(r^3 - 64) = 0 \Leftrightarrow$$

$$r^3 = 64 \Leftrightarrow r = 4\text{cm} (**)$$

Using  $(**)$  in  $(*)$ , we get  $h = 8 \text{ cm}$

$$\text{The sign of } \frac{dS}{dr} = \frac{4\pi(r^3 - 64)}{r^2} = \frac{4\pi(r - 4)(r^2 + 4r + 16)}{r^2}$$

depends on the sign of  $(r - 4)(r^2 + 4r + 16)$

Factors \ r	$-\infty$	4	$+\infty$
$r - 4$	-	0	+
$r^2 + 4r + 16$	+	+	+
$(r - 4)(r^2 + 4r + 16)$	-	0	+
S			

The surface is minimum at  $r = 4$  and  $h = 8 \text{ cm}$ .

Each tin should have a base radius of  $h = 4 \text{ cm}$  and perpendicular height  $h = 8 \text{ cm}$ .

6. Area  $A = \frac{1}{2} \times 6 \times 8 \times \sin \theta = 24 \sin \theta \text{ cm}^2$

$$\frac{dA}{d\theta} = 24 \cos \theta$$

When  $\theta = \frac{\pi}{4}$ ,  $\cos \theta = \frac{\sqrt{2}}{2}$

$$\frac{dA}{d\theta} = 24 \left( \frac{\sqrt{2}}{2} \right) = 12\sqrt{2} \text{ cm}^2 \text{ per radian}$$

**Task 9.11 on page 198**

1. (a) 5, 0

(b)  $6x - 6, 6$

(c)  $6x^2 - 10x + 4, 12x - 10$

(d)  $3x^2 + \frac{2}{x^2}, 6x - \frac{4}{x^3}$

(e)  $\frac{3}{2x^{\frac{1}{2}}} - \frac{1}{x^{\frac{3}{2}}}, -\frac{3}{4x^{\frac{3}{2}}} + \frac{3}{2x^{\frac{5}{2}}}$

(f)  $-\frac{6}{x^2} + \frac{6}{x^3} - \frac{12}{x^4}$  and  $\frac{12}{x^3} - \frac{18}{x^4} + \frac{48}{x^5}$

2. (a) Body changes direction at  $t = 2$

(b) Body does not change direction in the first 3 seconds.

3. 28 m

4. (a)  $(19.6 - 9.8t) \text{ ms}^{-1}, -9.8 \text{ ms}^{-2}$  (b) 2s

(c) 19.6 m (d) 0.586 s, 3.41s

**Task 9.12 on page 200**

1. Maximum profit is 12

2. Let the radius of the base of the cylinder be  $r$  and height  $h$ .

Thus,  $S = 2\pi rh + 2\pi r^2$  or  $h = \frac{S - 2\pi r^2}{2\pi r}$

$$V = \pi r^2 h = \pi r^2 \left( \frac{S - 2\pi r^2}{2\pi r} \right)$$

Volume,  $V = \frac{1}{2} (sr - 2\pi r^2)$

$$\frac{dV}{dr} = \frac{1}{2} (s - 6\pi r^2)$$

For maximum or minimum,

$$\frac{dV}{dr} = 0$$

$$\frac{1}{2} (s - 6\pi r^2) = 0$$

$$s - 6\pi r^2 = 0$$

$$r = \sqrt{\frac{s}{6\pi}}$$

$$\text{Note: } \frac{d^2V}{dr^2} = \frac{1}{2} (-12\pi r) = -6\pi r = -6\pi \sqrt{\frac{s}{6\pi}} = -\sqrt{6\pi s} \text{ (negative)}$$

$$\text{We have } h = \frac{s - 2\pi r^2}{2\pi r} = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r.$$

Hence, the height of the cylinder = diameter of the base

# Topic area: Linear algebra

UNIT

Sub-topic area: Vectors

# 10

## Vector space of real numbers

**Number of periods: 16**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define the scalar product of two vectors</li><li>• Give examples of scalar product</li><li>• Determine the magnitude of a vector and angle between two vectors</li></ul>	<ul style="list-style-type: none"><li>• Calculate the scalar product of two vectors</li><li>• Analyse a vector in terms of size.</li><li>• Calculate the angle between two vectors</li></ul>	<ul style="list-style-type: none"><li>• Apply and transfer the skills of dot product and magnitude to other areas of knowledge</li></ul>

### Content

1. Vector spaces  $\mathbb{R}^2$ :
  - Definitions and operations on vectors
  - Properties of vectors
  - Sub-vector spaces
2. Vector spaces of plane vectors  $\mathbb{R}, V, +$ :
  - Linear combination of vectors
  - Basis and dimension
3. Euclidian vector space:
  - Modulus (or magnitude) of vectors
  - Dot product and properties

### Materials required

Manila papers, graph papers, geometric instruments: ruler, T-square, protractors, computers.

## Generic competences

- Problem-solving
- Communication
- Cooperation
- Research

## Cross-cutting issues

- **Standardization culture**

Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

Groups consist of mixed gender and all are encouraged to participate.

## Teaching and learning activities

### Introductory activity

In groups, learners define what a vector is and use diagrams to differentiate between vectors and scalars. Let them do Task 10.1 on page 202 of the Student's Book.

### Main activities

1. Introduce the unit by defining the main concepts used in vectors. This to include the operations – addition, subtraction and multiplication involving vectors.
2. Guide students to define and understand the scalar product of two vectors. Use appropriate examples.
3. Guide learners on the operations and definitions involving vector spaces. Give them Task 10.2 on page 207 of the Student's Book to attempt in pairs.
4. Let them carry out research in advance on linear combination of vectors. Expound on spanning vectors and linear independent vectors. Let them do Task 10.3 on page 210.
5. Use examples to expound on linear dependent vectors.
6. Let students work in groups and research on the similarities between vector spaces and Euclidian spaces. Let them do Activity 10.1 on page 213 of the Student's Book.
7. Learners should be given a task to determine the magnitude and the angle between two vectors, plot these vectors and point out the dot product

of two vectors. Give them Activity 10.2 on page 217 and Activity 10.3 on page 218 to attempt in groups or pairs.

## Reinforcement activity

Learners discuss about the scalar product of two vectors.

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

### (b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

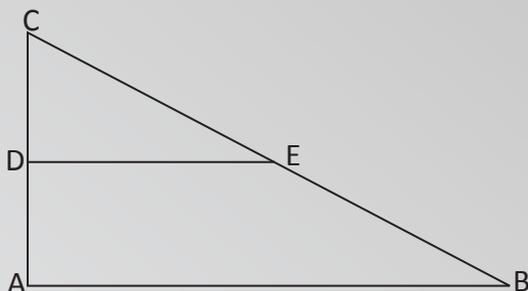
## Additional tasks

### Tasks 10 A - For slow learners

1. Show by using vectors that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is half of its length.
2. In a regular hexagon of vertices ABCDEF, prove that  $AB + AC + AD + AE + AF = 3AD$

### Answers

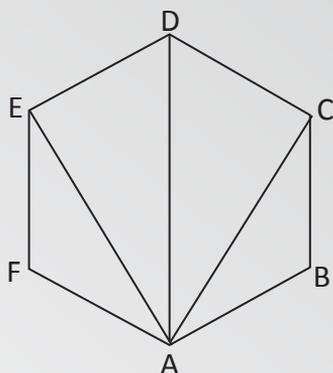
1. Let us illustrate the situation, where D and E are the mid-points of the sides AC and CB respectively.



We see that  $DE = DC + CE = \frac{1}{2}AC + \frac{1}{2}CB = \frac{1}{2}(AC + CB) = \frac{1}{2}AB$

Therefore,  $DE = \frac{1}{2}AB$

2. We make use of illustration below:



Since  $AB = ED$  and  $AF = CD$ , then

$$AB + AC + AD + AE + AF = ED + AC + AD + AE + CD = (AC + CD) + (AE + ED) + AD = AD + AD + AD = 3AD$$

Therefore,  $AB + AC + AD + AE + AF = 3AD$

### Task 10B - For talented learners

- Let  $\vec{e}_1 = (1, 0)$  and  $\vec{e}_2 = (0, 1)$ ; Show that  $\{\vec{e}_1, \vec{e}_2\}$  is a basis of vector space  $\mathbb{R}^2$ .
- Let  $\vec{u} = (1, -1)$  and  $\vec{v} = (3, 2)$ 
  - Show that  $S = \{\vec{u}, \vec{v}\}$  is a basis of  $\mathbb{R}^2$
  - Find  $\dim \mathbb{R}^2$ .

### Answers

- We have  $\vec{e}_1 = (1, 0)$  and  $\vec{e}_2 = (0, 1)$

The set  $\{\vec{e}_1, \vec{e}_2\}$  is a basis if it is independent and generating set  $\{e_1, e_2\}$  is linear independent if

$$\alpha \vec{e}_1 + \beta \vec{e}_2 = 0 \Rightarrow \alpha = \beta = 0$$

In fact;

$$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\alpha = \beta = 0$$

Thus,  $\{\vec{e}_1, \vec{e}_2\}$  is an independent set

$\{\vec{e}_1, \vec{e}_2\}$  is a generating set iff

$$\forall x, y \in \mathbb{R}^2, \exists \alpha, \beta \in \mathbb{R} : \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \vec{e}_1 + \beta \vec{e}_2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\alpha = x \quad \beta = y$$

Thus,  $\{\vec{e}_1, \vec{e}_2\}$  is a generating set.

Since  $\{\vec{e}_1, \vec{e}_2\}$  is linear independent and generating set. Hence we conclude that  $\{e_1, e_2\}$  is a basis of  $\mathbb{R}^2$ .

2. (a) We have to show that the vectors  $\vec{u} = (1, -1)$  and  $\vec{v} = (3, 2)$  are independent and generating vectors of  $\mathbb{R}^2$ .

We set  $\alpha(1, -1) + \beta(3, 2) = (0, 0)$  and we determine  $\alpha$  and  $\beta$  to see if the result will give us  $\alpha = \beta = 0$

$$\begin{cases} \alpha + 3\beta = 0 \\ -\alpha + 2\beta = 0 \end{cases}$$

$$\begin{cases} 5\beta = 0 \\ \alpha = -2\beta \end{cases}$$

$$\begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

$\beta = 0$  and  $\alpha = (0, 0)$  and . Thus the vectors  $\vec{u}$  and  $\vec{v}$  are independent.

We have to show that every vector in  $\mathbb{R}^2$  lies in the span of S. Indeed,

If  $(x, y) \in \mathbb{R}^2$  we must show that there exist scalars  $\alpha$  and  $\beta$  such

that  $(x, y) = \alpha(1, -1) + \beta(3, 2)$

$$\begin{cases} \alpha + 3\beta = x \\ -\alpha + 2\beta = y \end{cases}$$

$$\begin{cases} 5\beta = x + y \\ \alpha = 2\beta - y \end{cases}$$

$$\begin{cases} \beta = \frac{x+y}{5} \\ \alpha = x - 3\left(\frac{x+y}{5}\right) = \frac{5x - 3x - 3y}{5} = \frac{2x - 3y}{5} \end{cases}$$

$$\beta = \frac{x+y}{5} \text{ and } \alpha = \frac{2x-3y}{5}$$

This shows that  $\mathbb{R}^2 = \text{span } S$ .

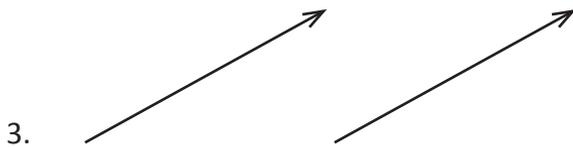
Hence, since  $S = \{\vec{u}, \vec{v}\}$  is independent and generating set of  $\mathbb{R}^2$  then it is the basis of  $\mathbb{R}^2$ .

- (b) Since the number of vectors in the basis of  $\mathbb{R}^2$  is 2 then,  $\dim \mathbb{R}^2 = 2$ .

## Answers to Tasks of Unit 10 in the Student's Book

### Task 10.1 on page 202

1. A vector is defined to be something that has magnitude and direction.
2. A quantity that is a real number, in other words, not a vector, is called a scalar. Unlike a vector, a scalar does not have direction.



### Task 10.2 on page 207

1. The proof is based on the properties of the vector space  $\mathbb{R}$ .
  - (a) We verify if  $(F(\mathbb{R}), +)$  is an abelian group, that is:
    - (i)  $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$  where we have used the fact that the addition of real numbers is commutative.
    - (ii)  $[(f + g) + h](x) = (f + g)(x) + h(x) = (fx) + g(x) + h(x)$   
 $= f(x) + (g(x) + h(x)) =$   
 $f(x) + (g + h)(x) = [f + (g + h)](x).$  (Associativity)
    - (iii) Let  $0$  be the zero function. Then for any  $f \in F(\mathbb{R})$  we have  
 $(f + 0)(x) = f(x) + 0(x) = f(x)$  (Additive identity is  $0$ )
    - (iv)  $[f + (-f)](x) = f(x) + (-f(x)) = f(x) - f(x) = 0.$   
 (Additive inverse of  $f(x)$  is  $-f(x)$ )

(b) We verify if the scalar multiplication satisfies:

(i)  $[\alpha(\beta f)](x) = \alpha[\beta f(x)] = \alpha\beta[f(x)] = [(\alpha\beta)f](x).$

(Associativity of multiplication)

(ii)  $(1f)(x) = 1f(x) = f(x).$

(Multiplicative identity is 1)

(iii)  $[\alpha(f + g)](x) = \alpha(f + g)(x) = \alpha f(x) + \alpha g(x) = (\alpha f + \alpha g)(x).$

(Right distributivity)

(iv)  $[(\alpha + \beta)f](x) = (\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x).$  (Left distributivity)

Thus,  $F(\mathbb{R})$  is a Vector Space.

2. For any  $(x, y) \in V$  with  $x, y > 0$  we have  $-(x, y) \notin V$ . The inverse element is not verified. Thus,  $V$  is not a vector space.

### Task 10.3 on page 210

1. Two vectors  $\vec{u}$  and  $\vec{v}$  are dependent if and only if one is a multiple of the other.

(a) No,

(b) yes, for  $\vec{v} = 3\vec{u}$

2. Since  $\alpha = \beta = 0$ ,  $\vec{i}$  and  $\vec{j}$  are linear independent.

3. (a) For any vector  $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$  in  $\mathbb{R}^2$ , we have  $c_1 = \left(\frac{5a - 2b}{7}\right)$  and  $c_2 = \frac{-4a + 3b}{7}$  for  $\vec{x} = c_1\vec{v} + c_2\vec{w}$ .

(b)  $c_1\vec{v} + c_2\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$  implies  $c_1 = c_2 = 0$ .

# Topic area: Linear algebra

## Sub-topic area: Linear transformation in 2D

# UNIT 11

## Concepts and operations on linear transformation in 2D

**Number of periods: 14**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define and distinguish between linear transformations in 2D</li><li>• Define central symmetry, orthogonal projection of a vector, identical transformation</li><li>• Define a rotation through an angle about the origin, reflection in the x axis, in y axis, in the line <math>y = x</math></li><li>• Show that a linear transformation is isomorphism in 2D or not</li></ul>	<ul style="list-style-type: none"><li>• Perform operations on linear transformations in 2D</li><li>• Construct the composite of two linear transformations in 2D</li><li>• Determine whether a linear transformation in 2D is isomorphism or not.</li><li>• Determine the analytic expression of the inverse of an isomorphism in 2D</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the importance and the use of operations on transformation in 2D</li><li>• Show curiosity for the study of operations on transformations in 2D</li></ul>

### Content

1. Linear transformations in 2D: Definition and properties
2. Geometric transformations: Definition and properties
3. Kernel and range
4. Operations of linear transformations

## Materials required

Manila papers, graph papers, Geometric instruments (ruler, pair of compasses, T-square), digital technology instruments including calculators

## Generic competences

- Research
- Cooperation
- Communication
- Problem-solving

## Cross-cutting issues

- **Standardization culture**

Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

Groups consist of mixed gender and all are encouraged to participate.

## Teaching and learning activities

### Introductory activity

Let students do Activity 11.1 on page 221 of the Student's Book.

### Main activities

1. Introduce the topic by defining a linear transformation in 2D. Guide them through definitions that are related to linear transformations.
2. Give them Task 11.1 on page 222 of the Student's Book to discuss whether each operation in 2D is linear or not.
3. Ask learners to carry out research and discussions on geometric transformation in 2D. Let them do Activity 11.2 on page 223 of the Student's Book. Take them through the transformations of rotation and reflection.
4. Help the students to determine whether a linear transformation in 2D is an isomorphism or not.
5. Guide them in determining the analytical expression of the inverse of an isomorphism in 2D. Let them tackle Task 11.2 on page 232 of the Student's Book.

6. Ensure they appreciate the importance and use of operations on transformation in 2D. Let them do Activity 11.3 on page 232 of the Student's Book. Give them Task 11.3 on page 236 to work on in pairs.

## Reinforcement activity

Learners should:

Perform operations on linear transformations in 2D

Solve  $f(v) = 0$  and determine kernel and range of  $f(v)$

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises.

Give them individual exercises to work on as homework.

### (b) Intellectual impairment

Try to understand the specific talents of the learner and develop them. Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task. Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

## Additional tasks

### Task 11A - For slow learners

1. A linear transformation  $T$  has matrix  $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ . Find
- the image of the point  $(2, 3)$  under  $T$ ,
  - the coordinates of the point having an image of  $(7, 2)$  under  $T$ .

### Answers

1. If  $(x', y')$  is the image of  $(x, y)$  then  $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

- (a) In this case  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2(2) - 1(3) \\ 1(2) + 1(3) \end{pmatrix} = \begin{pmatrix} 4 - 3 \\ 2 + 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Thus  $(1, 5)$  is the image of the point  $(2, 3)$  under  $T$ .

(b) In this case  $(x', y') = (7, 2)$

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$\begin{cases} 2x - y = 7 \\ x + y = 2 \end{cases}$$

$$x = 3 \text{ and } y = -1$$

Thus  $(3, -1)$  is the point having the image of  $(7, 2)$  under  $T$ .

### Task IIB - For talented learners

Show that the following mapping  $f$  is homomorphism

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ defined by } f(x, y) = (2x, x - y).$$

#### Answer

1.  $f(x, y) = (2x, x - y)$

We verify the two conditions of the definition.

For the first condition, we let  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $\mathbb{R}^2$  and compute

$$\begin{aligned} f((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = (2(x_1 + x_2), (x_1 + x_2) - (y_1 + y_2)) \\ &= (2x_1 + 2x_2, x_1 + x_2 - y_1 - y_2) \\ &= (2x_1, x_1 - y_1) + (2x_2, x_2 - y_2) \\ &= f(x_1, y_1) + f(x_2, y_2) \end{aligned}$$

This proves the first condition.

For the second condition, we let  $\alpha \in \mathbb{R}$  and  $(x, y) \in \mathbb{R}^2$  and we compute

$$f(\alpha(x, y)) = f(\alpha x, \alpha y) = (2\alpha x, \alpha x - \alpha y) = \alpha(2x, x - y) = \alpha f(x, y)$$

This proves the second condition

Since the two conditions are satisfied, we conclude that  $f$  is homomorphism (linear transformation)

### Assessment criteria

Demonstrate that a transformation of  $\mathbb{R}^2$  is linear. Perform operations on linear transformations.

## Answers to Tasks of Unit 11 in the Student's Book

### Task 11.1 on page 222

1. (a)  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2: f(x, y) = (y, 0)$

We verify the two conditions of the definition.

For the first condition, we let  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $\mathbb{R}^2$  and compute:

$$\begin{aligned} \mathbf{f}((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = (y_1 + y_2, 0) \\ &= (y_1, 0) + (y_2, 0) = \mathbf{f}(x_1, y_1) + \mathbf{f}(x_2, y_2) \end{aligned}$$

This proves the first condition.

For the second condition, we let  $\alpha \in \mathbb{R}$  and  $(x, y) \in \mathbb{R}^2$  and we compute

$$\mathbf{f}(\alpha(x, y)) = f(\alpha x, \alpha y) = (\alpha y, 0) = \alpha(y, 0) = \alpha \mathbf{f}(x, y).$$

This proves the second condition.

Since the two conditions are satisfied,  $f$  is a linear transformation.

(b)  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2: f(x, y) = (x^2, 0)$

We verify the two conditions of the definition.

For the first condition, we let  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $\mathbb{R}^2$  and compute:

$$\begin{aligned} \mathbf{f}((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = ((x_1 + x_2)^2, 0) \\ &\neq ((x_1)^2, 0) + ((x_2)^2, 0) = \mathbf{f}(x_1, y_1) + \mathbf{f}(x_2, y_2) \end{aligned}$$

This does not prove the first condition. There is no need to continue for the second condition.

Since the first condition is not verified, the given transformation  $f$  is not linear.

(c)  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2: f(x, y) = (x, 2x - y)$

We verify the two conditions of the definition.

For the first condition, we let  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $\mathbb{R}^2$  and compute:

$$\begin{aligned} \mathbf{f}((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, 2(x_1 + x_2) - (y_1 + y_2)) \\ &= x_1 + x_2, 2x_1 + 2x_2 - y_1 - y_2 \\ &= (x_1 + x_2, 2x_1 - y_1 + 2x_2 - y_2) \\ &= (x_1, 2x_1 - y_1) + (x_2, 2x_2 - y_2) \\ &= \mathbf{f}(x_1, y_1) + \mathbf{f}(x_2, y_2) \end{aligned}$$

This proves the first condition.

For the second condition, we let  $\alpha \in \mathbb{R}$  and  $(x, y) \in \mathbb{R}^2$  and we compute  $f(\alpha(x, y)) = f(\alpha x, \alpha y) = (\alpha x, 2\alpha x - \alpha y) = \alpha(x, 2x - y) = \alpha f(x, y)$ .

This proves the second condition

Since the two conditions are satisfied,  $f$  is a linear transformation.

(d)  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2: f(x, y) = (1, xy)$

We verify the two conditions of the definition.

For the first condition, we let  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $\mathbb{R}^2$  and compute:

$$\begin{aligned} f((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = (1, (x_1 + x_2)(y_1 + y_2)) \\ &= (1, x_1 y_1 + x_2 y_2 + x_1 y_2 + x_2 y_1) \\ &\neq (1, x_1 y_1) + (1, x_2 y_2) = \mathbf{f(x_1, y_1) + f(x_2, y_2)} \end{aligned}$$

This does not prove the first condition. There is no need to proceed to the second condition.

Since one condition is not verified, the given transformation  $f$  is not linear.

2.  $f(x, y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} (x, y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix} = (x + 2y, 3x + 4y)$

$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 (f(x, y) = (x + 2y, 3x + 4y))$

We verify the two conditions of the definition.

For the first condition, we let  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $\mathbb{R}^2$  and compute

$$\begin{aligned} \mathbf{f((x_1, y_1) + (x_2, y_2))} &= f(x_1 + x_2, y_1 + y_2) \\ &= ((x_1 + x_2) + 2(y_1 + y_2), 3(x_1 + x_2) + 4(y_1 + y_2)) \\ &= x_1 + x_2 + 2y_1 + 2y_2, 3x_1 + 3x_2 + 4y_1 + 4y_2 \\ &= (x_1 + 2y_1, 3x_1 + 4y_1) + (x_2 + 2y_2, 3x_2 + 4y_2) \\ &= \mathbf{f(x_1, y_1) + f(x_2, y_2)} \end{aligned}$$

This proves the first condition.

For the second condition, we let  $\alpha \in \mathbb{R}$  and  $(x, y) \in \mathbb{R}^2$  and we compute

$$\begin{aligned} \mathbf{f(\alpha(x, y))} &= f(\alpha x, \alpha y) = (\alpha x + 2\alpha y, 3\alpha x + 4\alpha y) = \\ &\alpha(x + 2y, 3x + 4y) = \mathbf{\alpha f(x, y)}. \end{aligned}$$

This proves the second condition.

Since the two conditions are satisfied,  $f$  is a linear transformation.

### Task 11.2 on page 232

$L = \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $L(x, y) = \begin{bmatrix} 3x + 2y \\ -x + y \end{bmatrix}$ .

$\ker f = \{v \in \mathbb{R}^2 : f(v) = 0 \in \mathbb{R}^2\} = \{(x, y) \in \mathbb{R}^2 : (3x + 2y, -x + y) = (0, 0)\}$

$$= \left\{ (x, y) \in \mathbb{R}^2 : \begin{cases} 3x + 2y = 0 \\ -x + y = 0 \end{cases} \right\} = \left\{ (x, y) \in \mathbb{R}^2 : \begin{cases} 3x + 2y = 0 \\ 2x - 2y = 0 \end{cases} \right\}$$

$$= \left\{ (x, y) \in \mathbb{R}^2 : \begin{cases} 5x = 0 \\ x = y \end{cases} \right\} = \{(x, y) \in \mathbb{R}^2 : x = y = 0\} = \{(0, 0) \in \mathbb{R}^2\}$$

$\ker f = \{(0, 0) \in \mathbb{R}^2\}$ .

Thus,  $\dim \ker f = 0$

### Task 11.3 on page 236

1.  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2: f(x, y) = (3x - y, 2x + y)$

We have to verify if  $f$  is one-to-one and onto.

Let  $u = (x_1, x_2)$  and  $v = (y_1, y_2)$ , then if  $f(u) = f(v)$ , we have

$(3x_1 - x_2, 2x_1 + x_2) = (3y_1 - y_2, 2y_1 + y_2)$ . We have the system

$$\begin{cases} 3x_1 - x_2 = 3y_1 - y_2 \\ 2x_1 + x_2 = 2y_1 + y_2 \end{cases} \text{ solving the system, we have } \begin{cases} 5x_1 = 5y_1 \\ 2x_1 + x_2 = 2y_1 + y_2 \end{cases} \Leftrightarrow$$

$$\begin{cases} x_1 = y_1 \\ 2x_1 + x_2 = 2y_1 + y_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_2 \end{cases}$$

Thus,  $f$  is one-to-one.

Let  $(a, b)$  be an arbitrary element in the range space, then we have to verify if the following system has solution for all values of  $a$  and  $b$ .

$$\begin{cases} 3x - y = a \\ 2x + y = b \end{cases} \Leftrightarrow \begin{cases} 5x = a + b \\ y = b - 2x \end{cases} \Leftrightarrow \begin{cases} x = \frac{a + b}{5} \\ y = b - 2\left(\frac{a + b}{5}\right) \end{cases} \Leftrightarrow \begin{cases} x = \frac{a + b}{5} \\ y = \frac{5b - 2a - 2b}{5} \end{cases} \Leftrightarrow$$

$$\begin{cases} x = \frac{a + b}{5} \\ y = \frac{3b - 2a}{5} \end{cases} \text{ . We have solution for } x \text{ and } y.$$

Thus, the transformation is onto.

Hence, the transformation  $f$  is isomorphism.

2.  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2: f(x, y) = (2x - 4y, -3y + 6y)$

We have to verify if  $f$  is one-to-one and onto.

Let  $u = (x_1, x_2)$  and  $v = (y_1, y_2)$ , then if  $f(u) = f(v)$ , we have

$(2x_1 - 4x_2, -3x_1 + 6x_2) = (2y_1 - 4y_2, -3y_1 + 6y_2)$  we have the system

$$\begin{cases} 2x_1 - 4x_2 = 2y_1 - 4y_2 \\ -3x_1 + 6x_2 = -3y_1 + 6y_2 \end{cases}$$

solving the system, we have

$$\begin{cases} 6x_1 - 12x_2 = 6y_1 - 12y_2 \\ -6x_1 + 12x_2 = -6y_1 + 12y_2 \end{cases}$$

$$0x_1 + 0x_2 = 0y_1 + 0y_2.$$

Some different vectors of the domain have the same image.

Thus  $f$  is not one-to-one. There is no need to verify if the transformation is onto.

Hence, the transformation  $f$  is not isomorphism.

# Topic area: Linear algebra

## Sub-topic area: Linear transformation in 2D

# UNIT 12 Matrices and determinants of order 2

**Number of lessons: 12**

### **Learning objectives**

By the end of this unit, the student should be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define the order of a matrix</li><li>• Define a linear transformation in 2D by a matrix</li><li>• Define operations on matrices of order 2</li><li>• Show whether a square matrix of order 2 is invertible or not</li></ul>	<ul style="list-style-type: none"><li>• Reorganise data into matrices</li><li>• Determine the matrix of a linear transformation in 2D</li><li>• Perform operations on matrices of order 2</li><li>• Construct the matrix of the composite of two linear transformations in 2D</li><li>• Construct the matrix of the inverse of an isomorphism of <math>\mathbb{R}^2</math></li><li>• Determine the inverse of a matrix of order 2</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the importance and the use of matrices in organising data</li><li>• Show curiosity for the study of matrices of order 2</li></ul>

### **Content**

1. Matrix of a linear transformation: Definition and operations
2. Matrix of geometric transformation
3. Operations on matrices:
  - Equality of matrices

- Addition
  - Multiplication of matrices
  - Transpose of a matrix
  - Inverse of a matrix
4. Determinant of a matrix of order 2:
- Definition
  - Applications of determinants

## Materials required

Manila papers, markers,

## Generic competences

- Critical thinking
- Cooperation
- Communication
- Research

## Cross-cutting issues

- **Standardization culture**

Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

Groups consist of mixed gender and all are encouraged to participate.

## Teaching and learning activities

### Introductory activity

Let them tackle Activity 12.1 on page 238 of the Student's Book to research on and explain what a matrix is.

### Main activities

1. Guide them to define the order of a matrix. Let them also reorganise data into matrices. Let them be able to appreciate the importance and use of matrices in organising data.
2. Expound on the unit by defining a linear transformation by matrix in 2D.
3. Give them the Activity 12.2 on page 241 to carry out.

4. Guide them in carrying out operations on matrices – addition, subtraction and multiplication. Let them do Activity 12.3 on page 241 then Task 12.1 on page 242 and Task 12.2 on page 244 of the Student’s Book.
5. Explain to them what a transpose matrix is and let them attempt Task 12.3 on page 246 of the Student’s Book.
6. Give them Activity 12.4 on page 246 to find out what a determinant of a matrix is.
7. Help learners to determine the inverse of a matrix of order 2. Learners in groups to show whether or not a matrix of order 2 is invertible. Let them attempt Task 12.4 on page 248 of the Student’s Book.
8. Let them practise on uses of matrices by doing Task 12.5 on page 250.

### Reinforcement activity

Learners should carryout research on the importance and use of matrices, for example in physics, economics, entrepreneurship and sport and report the findings.

### Learners of varying strengths and abilities

#### (a) Gifted and talented

You can provide more advanced material to the learner. Allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

#### (b) Slow learners

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

### Additional tasks

#### Task I2A - For slow learners

Find  $x$ ,  $y$ ,  $z$  and  $t$  if  $3 \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2t \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+t & 3 \end{pmatrix}$

## Answers

$$3 \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2t \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+t & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3x & 3y \\ 3z & 3t \end{pmatrix} = \begin{pmatrix} x+4 & 6+x+y \\ -1+z+1 & 2t+3 \end{pmatrix}$$

We form the system of equation below and solve

$$\begin{cases} 3x = x + 4 \\ 3y = 6 + x + y \\ 3z = -1 + z + t \\ 3t = 2t + 3 \end{cases} \quad \begin{cases} x = 2 \\ y = \frac{11}{2} \\ z = \frac{-1+1}{2} \\ t = 3 \end{cases} \quad \begin{cases} x = 2 \\ y = \frac{11}{2} \\ z = 1 \\ t = 3 \end{cases}$$

Therefore,  $x = 2$ ,  $y = \frac{11}{2}$ ,  $z = 1$  and  $t = 3$

## Task 12B - For talented learners

1. Given that  $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ ,  $f(x) = 2x^2 - 3x + 5$  and  $g(x) = x^2 + 3x - 10$ . Determine:
- (a)  $A^2$  (c)  $f(A)$   
(b)  $A^3$  (d)  $g(A)$

## Answers

$$1. \quad (a) \quad A^2 = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 1+6 & 2-8 \\ 3-12 & 6+16 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$$

$$(b) \quad A^3 = A^2 A = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 7-18 & 14+24 \\ -9+66 & -18-88 \end{pmatrix} \\ = \begin{pmatrix} -11 & 38 \\ 57 & -106 \end{pmatrix}$$

$$(c) \quad f(A) = 2A^2 - 3A + 5I = 2 \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} - 3 \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 14 & -12 \\ -18 & 44 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 9 & -12 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 14-3+5 & -12-6+0 \\ -18-9+0 & 44+12+5 \end{pmatrix} \\ = \begin{pmatrix} 16 & -18 \\ -27 & 61 \end{pmatrix}$$

$$(d) \quad g(A) = A^2 + 3A - 10I = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} + 3 \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - 10 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 9 & -12 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 7+3-10 & -6+6-0 \\ -9+9-0 & 22-12-10 \end{pmatrix} \\ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

we can say that a matrix is a zero (root) of the polynomial  $g(x)$ .

## Assessment criteria

Use matrices and determinants of order 2 to solve systems of linear equations and to define transformations.

## Answers to Tasks of Unit 12 in the Student's Book

### Task 12.1 on page 242

$$x = 4, y = -6 \text{ and } z = 9$$

### Task 12.2 on page 244

$$a) \begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -1 & 0 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -7 & 4 \end{bmatrix}$$

$$c) \begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ -6 & 2 & 9 \end{bmatrix}$$

$$d) \begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -9 & 2 \\ 4 & 6 & 3 \end{bmatrix}$$

### Task 12.3 on page 246

$$a) AB = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 10 & 22 \end{bmatrix}$$

$$b) AB = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 16 \end{bmatrix}$$

$$c) QP = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$$

$$d) BA = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ -1 & 4 & -2 \end{bmatrix} \times \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 4 \\ 1 & 3 & -17 \\ -11 & -10 & 31 \end{bmatrix}$$

### Task 12.4 on page 248

$$1. \text{ If } A = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}, \text{ then } 2A = 2 \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 8 & 12 \end{pmatrix}$$

$$-A = -\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & -6 \end{pmatrix}$$

$$\frac{1}{2}A = \frac{1}{2} \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 3 \end{pmatrix}$$

$$2. \quad AB = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 6-6 & 12-12 \\ 12-12 & 24-24 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$3. \text{ If } B = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}, \text{ then } B^0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1-0 & -2-6 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 0 & 9 \end{pmatrix}$$

$$B^3 = B^2B = \begin{pmatrix} 1 & -8 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1-0 & -2-24 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & -26 \\ 0 & 27 \end{pmatrix}.$$

4.  $\det A = 2$
5.  $\det D = 0$ . D is a singular matrix
6.  $AB = \begin{pmatrix} 4 & 5 \\ -1 & 4 \end{pmatrix} \neq \begin{pmatrix} 8 & 7 \\ -3 & 0 \end{pmatrix} = BA$
7. Since  $A$  does not have an inverse. Since  $B = 1 \neq 0$  has an inverse  $B^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ .

**Task 12.5 on page 250**

1. 30,000 FRW for Panadol tin      12,000 FRW for Fansidar tin.
2.  $x = 12$  cm     $y = 6$  cm
3. The costs of a bacon and a sausage are 4,700FRW and 1,100FRW respectively.
4. Airfares for first class and third class are 25,000 FRW and 14,5000 FRW respectively.

# Topic area: Geometry

## Sub-topic area: Plane geometry

# UNIT 13

## Points, straight lines and circles in 2D

**Number of periods: 21**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"> <li>Define the coordinate of a point in 2-D</li> <li>Define a straight line knowing its:               <ul style="list-style-type: none"> <li>- 2 points</li> <li>- direction vector</li> <li>- gradient</li> </ul> </li> </ul>	<ul style="list-style-type: none"> <li>Represent a point and or a vector in 2D - calculate the distance between two points in 2D and the mid-point of a segment in 2D</li> <li>Determine equations of a straight line (vector equation, parametric equation, Cartesian equation)</li> <li>Apply knowledge to find the centre, radius, and diameter to find the equation of a circle</li> <li>Perform operations to determine the intersection of a circle and a line</li> </ul>	<ul style="list-style-type: none"> <li>Appreciate that a point is a fixed position in a plane</li> <li>Show concern patiently and mutual respect in representations and calculations</li> <li>Be accurate in representations and calculations</li> <li>Manifest a team spirit and think critically in problem solving related to the positions of straight lines in 2D</li> </ul>

### Content

- Points in 2D:
  - Cartesian coordinates of a point
  - Distance between two points
  - Mid-point of a line segment

2. Lines in 2D:
  - Equations of a line
  - Vector equation
  - Parametric equations
  - Cartesian equation
3. Problems on points and straight lines in 2D
  - Position, angle, distance
4. The circle
  - Cartesian equation of a circle
  - Problems involving position of a circle and point or position of a circle and lines in 2D

## Materials required

Manila paper, graph paper, geometric instruments (ruler, T-square), digital technology equipment including calculators.

## Generic competences

- Critical thinking
- Cooperation
- Communication
- Research

## Cross cutting issues

- **Standardization culture**

Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

Groups consist of mixed gender and all are encouraged to participate.

## Teaching and learning activities

### Introductory activity

Ask students to explain what a point is. They should also explain what a Cartesian plane is. Let them define the coordinate of a point in 2D. Let them do Activity 13.1 on page 252 of the Student's Book.

## Main activities

1. Revise with learners how to get the distance between two points in the Cartesian plane. Also guide them in locating the mid-point of a line. Then let them work in pairs or small groups to tackle Task 13.1 on page 255 of the Student's Book.
2. Guide them to determine algebraic representations of straight lines in 2D. Let them work on Task 13.2 and Task 13.3 on pages 258 and 260 respectively of the Student's Book.
3. In groups learners discuss the distance between two vectors.
4. Learners represent on graph papers some chosen points, lines and or circles and determine their parametric or Cartesian equations. Give them Task 13.4 on page 263 to tackle in pairs, or groups.
5. Assist them to represent the equations of a straight line in 2D.
6. Let them ponder over the Mental task on page 263 of the Student's Book to recall the main points and parts of a circle.
7. Ask them to research on what a unit circle is and to present their findings in a discussion in class. Allow them to explain with examples on the chalk board.
8. Expound on the concept of intersection of lines and circles then on collinear points. Give them Task 13.5 on page 267 of the Student's Book.

## Reinforcement activities

Determine algebraic representations of lines, straight lines and circles in 2D.

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner. For example, let them draw extra graphs and allow them to do extra exercises. You can also make them group leaders in their groups so that they indirectly help the slower learners or the not so talented. However, ensure that they are supervised so as not to overwhelm the less gifted.

Give them individual exercises to work on as homework.

**(b) Slow learners**

You should provide them with the non-restrictive environment that provides/maintains their degree of freedom, self-determination, dignity and integrity of mind and body. This should, however allow for participation in an inclusive setting. Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

## Additional tasks

### Task I3A - For slow learners

1. Let L be the line that passes through the points A(1, 3) and B(2, -4). Find an equation for the line L.

**Answer**

1. We are given points A(1, 3) and (2, -4)

$$L \equiv y - 3 = \frac{3+4}{1-2}(x - 1)$$

$$L \equiv y - 3 = \frac{7}{-1}(x - 1)$$

$$L \equiv y = -7x + 10$$

$$L \equiv y = 10 - 7x$$

### Task I3B - For talented learners

1. If one end of a diameter of the circle  $x^2 + y^2 - 4x - 6y + 11 = 0$  is (3, 4), then find the other end of the diameter on the circle.

**Answer**

1. Equation of the circle is  $x^2 + y^2 - 4x - 6y + 11 = 0$

$$x^2 - 4x + y^2 - 6y = -11$$

By completing squares

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) = -11 + 4 + 9$$

$$(x - 2)^2 + (y - 3)^2 = 2$$

The centre of the circle is (2, 3) and the radius is  $\sqrt{2}$ .

That centre is the midpoint of the diameter.

Let the other end point of the diameter be (a, b)

$$(2, 3) = \left(\frac{3+a}{2}, \frac{4+b}{2}\right)$$

We have the system of two equations with two unknowns

$$\begin{cases} 3 + a = 4 \\ 4 + b = 6 \end{cases}$$

$$\begin{cases} a = 1 \\ b = 2 \end{cases}$$

$$\begin{cases} \frac{3+a}{2} = 2 \\ \frac{4+b}{2} = 3 \end{cases}$$

The other end of the diameter on the circle is (1, 2).

## Assessment criteria

Determine algebraic representations of lines and circles in 2D.

## Answers to Tasks of Unit 13 in the Student's Book

### Task 13.1 on page 255

1. (a) (6, 5) (b) (3, 1) (c)  $(\frac{7}{2}, 5)$   
(d) (1, 3) (e) (-5, -5) (f) (4, 2)  
(g) (-9, 4)

2. (a)  $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$  (b)  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  (c)  $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$   
d)  $3\vec{i}$  (e)  $6\vec{i} + 2\vec{j}$  (f)  $-4\vec{i} - 2\vec{j}$

3. B (2, 3)

4.  $P = 5$  and  $q = -1$

5. E(-9, -8)

6. (a) (3, 1) (b) (6, 5) (c) 5

7. If we have A(9, 9), B(3, 2) and C(9, 4)

(a) The coordinates of M, the midpoint of BC is  $\frac{1}{2}BC = \left(\frac{3+9}{2}, \frac{2+4}{2}\right) = (6, 3)$

(b) The length of the median from A to M is

$$|AM| = \sqrt{(6-9)^2 + (3-9)^2} = \sqrt{9+36} = \sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$$

8. A(0, 1), B(2, 7) and C(4, -1)

The midpoints are

$$M_{BC} = \left(\frac{2+4}{2}, \frac{7-1}{2}\right) = (3, 3)$$

$$M_{AB} = \left(\frac{0+2}{2}, \frac{1+7}{2}\right) = (1, 4), \text{ and}$$

$$M_{AC} = \left( \frac{0+4}{2}, \frac{1-1}{2} \right) = (2, 0)$$

The lengths of the medians are

$$|AM_{BC}| = \sqrt{(3-0)^2 + (3-1)^2} = \sqrt{9+4} = \sqrt{13}$$

$$|CM_{AB}| = \sqrt{(1-4)^2 + (4+1)^2} = \sqrt{9+25} = \sqrt{34}$$

$$|BM_{AC}| = \sqrt{(2-2)^2 + (0-7)^2} = \sqrt{0+49} = 7$$

9. We are given the points of a parallelogram  $P(-1, 5)$ ,  $Q(8, 10)$ ,  $R(7, 5)$  and  $S$ . We have to find the coordinates of  $S$ .

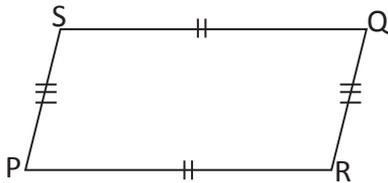
$$|PQ| = \sqrt{(8+1)^2 + (10-5)^2} = \sqrt{81+25} = \sqrt{106}$$

$$|QR| = \sqrt{(7-8)^2 + (5-10)^2} = \sqrt{1+25} = \sqrt{26}$$

$$|PR| = \sqrt{(7+1)^2 + (5-5)^2} = \sqrt{64+0} = \sqrt{64} = 8$$

Since  $|PQ|$  is longer, it is a diagonal.

We can illustrate this as below:



We have  $P(-1, 5)$ ,  $Q(8, 10)$ ,  $R(7, 5)$ .

We find the gradient of the line  $RQ$  which is the same as the one of the line  $PS$  and is  $\frac{10-5}{8-7}$

The equation of the line  $PS$  is

$$y - 5 = 5(x + 1)$$

$$y = 5x + 10$$

We find the gradient of the line  $PR$  which is the same as the one of the line  $QS$  and is  $\frac{5-5}{7+1} = 0$

The equation of the line  $QS$  is

$$y - 10 = 0(x - 8)$$

$$y = 10$$

We have to find the point  $S$  which is the intersection of the lines

$y = 5x + 10$  and  $y = 10$ . Therefore, to find  $S$ , we have to solve the following system:

$$\begin{cases} y = 5x + 10 \\ y = 10 \end{cases}$$

$$\begin{cases} 10 = 5x + 10 \\ y = 10 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 10 \end{cases}$$

The coordinates of the point S are (0, 10).

10. We are given the points of a parallelogram A(1, 1), B(2, 7), C(13, 7) and D.

(a) We have to find the coordinates of D.

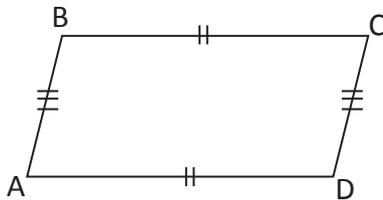
$$|AB| = \sqrt{(2-1)^2 + (7-1)^2} = \sqrt{1+36} = \sqrt{37}$$

$$|AC| = \sqrt{(13-1)^2 + (7-1)^2} = \sqrt{144+36} = \sqrt{180}$$

$$|BC| = \sqrt{(13-2)^2 + (7-7)^2} = \sqrt{121+0} = \sqrt{121} = 11$$

Since  $|AC|$  is longer, it is a diagonal.

We can illustrate this as below:



We have A(1, 1), B(2, 7), C(13, 7).

We find the gradient of the line AB which is the same as the one of the line CD and is  $\frac{7-1}{2-1} = 6$

The equation of the line CD is

$$y - 7 = 6(x - 13)$$

$$y = 6x - 71$$

We find the gradient of the line BC which is the same as the one of the line AD and is  $\frac{7-7}{13-2} = 0$

The equation of the line AD is

$$y - 1 = 0(x - 1)$$

$$y = 1$$

We have to find the point D which is the intersection of the lines  $y = 6x - 71$  and  $y = 1$ . Therefore, to find D, we have to solve the following system:

$$\begin{cases} y = 6x - 71 \\ y = 1 \end{cases}$$

$$\begin{cases} 1 = 6x - 71 \\ y = 1 \end{cases}$$

$$\begin{cases} x = 12 \\ y = 1 \end{cases}$$

The coordinates of the point D are (12, 1).

(b) The coordinates of the midpoint of the diagonal DB is

$$\left( \frac{12+2}{2}, \frac{7+1}{2} \right) = (7, 4)$$

(c) The coordinates of the midpoint of the diagonal AC is

$$\left( \frac{1+13}{2}, \frac{1+7}{2} \right) = (7, 4)$$

### **Task 13.2 on page 258**

1. (a) (1, 2) and (0, 2)

$$y - 2 = \frac{2-2}{1-0}(x - 1)$$

$$y - 2 = 0(x - 1)$$

$$y = 2$$

(b) (1, 3) and (-2, 5)

$$y - 3 = \frac{3-5}{1+2}(x - 1)$$

$$y = -\frac{2}{3}(x - 1) + 3$$

$$y = -\frac{2}{3}x + \frac{11}{3}$$

(c) (-3, 0) and (1, 4)

$$y - 0 = \frac{4-0}{1+3}(x + 3)$$

$$y = 1(x + 3)$$

$$y = x + 3$$

(d) (8, 2) and (3, -5)

$$y - 2 = \frac{2 + 5}{8 - 3}(x - 8)$$

$$y = \frac{7}{5}(x - 8) + 2$$

$$y = \frac{7}{5}x - \frac{46}{5}$$

(e) (4, -2) and (6, -1)

$$y + 2 = \frac{-1 + 2}{6 - 4}(x - 4)$$

$$y = \frac{1}{2}(x - 4) + 2$$

$$y = \frac{1}{2}x - 4$$

2.

(a) (2, 3) and gradient is 2

$$y - 3 = 2(x - 2)$$

$$y = 2x - 1$$

(b) (1, 5) and gradient is -1

$$y - 5 = -1(x - 1)$$

$$y = -x + 6$$

(c) (-3, 0) and gradient is 3

$$y - 0 = 3(x + 3)$$

$$y = 3x + 9$$

(d) (4, 2) and gradient is 1

$$y - 2 = 1(x - 4)$$

$$y = x - 2$$

(e) (9, -3) and gradient is -2

$$y + 3 = -2(x - 9)$$

$$y = -2x + 15$$

3. C(1, 3), D(4, 2), E(-3, -1) and F(-1, 5)

$$M_{CD} = \left( \frac{1+4}{2}, \frac{3+2}{2} \right) = \left( \frac{5}{2}, \frac{5}{2} \right)$$

$$M_{EF} = \left( \frac{-3-1}{2}, \frac{-1+5}{2} \right) = (-2, 2)$$

$$y - 2 = \frac{\frac{5}{2} - 2}{\frac{5}{2} + 2}(x + 2)$$

$$y - 2 = \frac{\frac{1}{2}}{\frac{9}{2}}(x + 2)$$

$$y = \frac{2}{9}(x + 2) + 2$$

$$y = \frac{2}{9}x + \frac{22}{9}$$

4. A(2, 4) and B(-1, 3)

Gradient of AB is  $\frac{4-3}{2+1} = \frac{1}{3}$

The required line is perpendicular to the line AB and its gradient is  $-3$ .

The required line passes through the midpoint of AB which is  $\left(\frac{2-1}{2}, \frac{4+3}{2}\right) = \left(\frac{1}{2}, \frac{7}{2}\right)$ .

The equation of the line is

$$y - \frac{7}{2} = -3\left(x - \frac{1}{2}\right)$$

$$y = -3x + 5$$

5. A(3, 1) and B(4, 8)

(a) The mid-point of AB is  $\left(\frac{3+4}{2}, \frac{1+8}{2}\right) = \left(\frac{7}{2}, \frac{9}{2}\right)$

(b) The gradient of AB is  $\frac{8-1}{4-3} = 7$

(c) The required line is perpendicular to the line AB and its gradient is  $-\frac{1}{7}$

The required line passes through the midpoint of AB which is  $\left(\frac{7}{2}, \frac{9}{2}\right)$ .

The equation of the line is

$$y - \frac{9}{2} = -\frac{1}{7}\left(x - \frac{7}{2}\right)$$

$$y = -\frac{1}{7}x + 5$$

6. A(0, 7), B(9, 4) and C(1, 0)

(a) The gradient of the line AB is  $\frac{4-7}{9-0} = -\frac{1}{3}$

The required line is perpendicular to the line AB and its gradient is 3.

The required line passes through the point C(1, 0)

The equation of the required line is

$$y - 0 = 3(x - 1)$$

$$y = 3x - 3$$

(b) The required line passes through the point A(0, 7) and the midpoint of BC which is  $\left(\frac{9+1}{2}, \frac{4+0}{2}\right) = (5, 2)$ . The equation of the required line is

$$y - 2 = \frac{7-2}{0-5}(x - 5)$$

$$y - 2 = -1(x - 5)$$

$$y = -x + 7$$

7. A(1, 0), B(5, 2) and C(1, 6)

The midpoints are:

$$M_{BC} = \left(\frac{5+1}{2}, \frac{2+6}{2}\right) = (3, 4)$$

$$M_{AC} = \left(\frac{1+1}{2}, \frac{0+6}{2}\right) = (1, 3)$$

$$M_{AB} = \left(\frac{1+5}{2}, \frac{0+2}{2}\right) = (3, 1)$$

The median from A to BC has equation

$$y - 4 = \frac{4-0}{3-1}(x - 3)$$

$$y - 4 = 2(x - 3)$$

$$y = 2x - 2$$

The median from B to AC has equation

$$y - 3 = \frac{3-2}{1-5}(x - 1)$$

$$y - 3 = -\frac{1}{4}(x - 1)$$

$$y = -\frac{1}{4}x + \frac{13}{4}$$

The median from C to AB has equation

$$y - 1 = \frac{6-1}{1-3}(x - 3)$$

$$y = -\frac{5}{2}(x - 3) + 1$$

### **Task 13.3 on page 260**

(a) (4, -1)

(b) (2, 1)

(c) (2, 3)

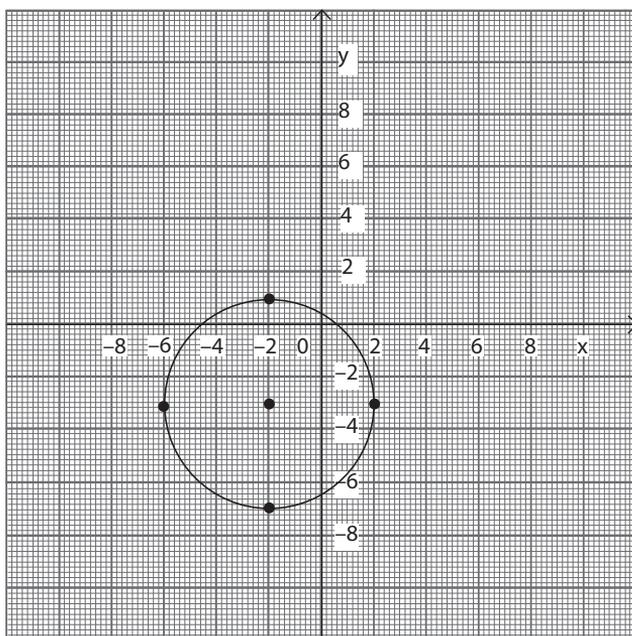
(d) (3, 5)

**Task 13.4 on page 263**

- (a)  $\frac{1}{3}$       (b)  $\frac{3}{4}$       (c)  $5\frac{1}{2}$
- $26.6^\circ$ ,  $45^\circ$ ,  $108.4^\circ$

**Task 13.5 on page 267**

- $x^2+y^2+22x-18y+186=0$
- $(x+7)^2+(y-11)^2=317$
- 6
- $(-4, 6)$
- The centre is  $(-2, -3)$  and the radius is 5 units of length.



# Topic area: Statistics and probability

UNIT Sub-topic area: Descriptive statistics

## 14 Measures of dispersion

**Number of periods: 7**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define the variance, standard deviation and the coefficient of variation</li><li>• Analyse and interpret critically data and infer conclusion.</li></ul>	<ul style="list-style-type: none"><li>• Determine the measures of dispersion of a given statistical series.</li><li>• Apply and explain the standard deviation as the more convenient measure of the variability in the interpretation of data</li><li>• Express the coefficient of variation as a measure of the spread of a set of data as a proportion of its mean.</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the importance of measures of dispersion in the interpretation of data</li><li>• Show concern on how to use the standard deviation as measure of variability of data.</li></ul>

### Content

1. Introduction - measures of central tendency
2. Measures of dispersion
3. Coefficient of variation

- Problems to include measure of dispersion and explain the standard deviation as the more convenient measure of variability in the interpretation of data.
- Problems to include measure of dispersion and express the coefficient of variation as a measure of the spread of a set of data as a proportion of its mean.

## Materials required

Manila papers, graph papers, ruler, digital technology including calculators.

## Generic competences

- Critical thinking
- Problem solving
- Cooperation
- Communication

## Cross-cutting issues

- **Financial education**

Use of examples that encourage planning for and prudent use of money.

- **Standardization culture**

Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

Groups consist of mixed gender and all are encouraged to participate.

## Teaching and learning activities

### Introductory activity

Revise with learners what they learnt in Junior Secondary statistics. Let them work on Activity 14.1 on page 269 to refresh their understanding on the measures of central tendency.

### Main activities

1. In groups, learners will discuss the measures of dispersion, interpret them and represent their findings. Let them do Activity 14.2 on page 275 of the Student's Book.

2. Discuss and explain the range, to include the interquartile range and the semi-interquartile range.
3. Explain variability of data.
4. Introduce them to standard deviation and variance. Use a number of examples. Encourage learners to work out some examples on the chalkboard for all to discuss. Give them Task 14.1 on page 281 of the Student's Book to work out.
5. Let students research on the meaning of coefficient of variation. Discuss their findings to come up with a concise definition.
6. Give them Activity 14.3 on page 282 to research on and discuss real life applications of measures of dispersion. Let them do Task 14.2 on page 283 of the Student's Book.
7. Help them appreciate the importance of measures of dispersion.

### **Reinforcement activities**

Extend understanding, analysis and interpretation of data arising from problems and questions in daily life to include the standard deviation.

### **Learners of varying strengths and abilities**

#### **(a) Physical impairment**

- Use cooperative learning for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so as to be helped.
- Provide these students with frequent progress checks.

#### **(b) Intellectual impairment**

- Try to understand the specific talents of the learner and develop them.
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task.
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

## Additional tasks

### Task I4A - For slow learners

1. The numbers of incorrect answers on a true-false competency test for a random sample of 15 students were recorded as follows:

2, 1, 3, 0, 1, 3, 6, 0, 3, 3, 5, 2, 1, 4, and 2.

Find the mean, the median, the mode and the sample standard deviation.

#### Answers

1. The mean is

$$\begin{aligned}\bar{x} &= \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{15} (2 + 1 + 3 + 0 + 1 + 3 + 6 + 0 + 3 + 3 + 5 + 2 + 1 + 4 + 2) \\ &= \frac{1}{15} (36) = 2.4\end{aligned}$$

The median is the middle observation in the ascending data:

0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 5, 6. So, the median is 2.

The mode is the most frequently occurring observation in a sample. So the mode is 3.

The sample standard deviation is  $\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$ . We need a table:

N <sup>o</sup>	$x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1	0	-2.4	5.76
2	0	-2.4	5.76
3	1	-1.4	1.96
4	1	-1.4	1.96
5	1	-1.4	1.96
6	2	-0.4	0.16
7	2	-0.4	0.16
8	2	-0.4	0.16
9	3	0.6	0.36
10	3	0.6	0.36
11	3	0.6	0.36
12	3	0.6	0.36
13	4	1.6	2.56
14	5	2.6	6.76
15	6	3.6	12.96

$$\sum_{i=1}^n (x_i - \bar{x})^2 = 41.6$$

$$\sigma = \sqrt{\frac{1}{15} \sum_{i=1}^n (x_i - \bar{x})^2} = \sqrt{\frac{1}{15}(41.6)} = \sqrt{2.77} = 1.66$$

### Task 14B - For talented learners

1. If the sum of the first  $n$  natural numbers and the sum of the first square natural numbers is given by

$$\sum x_i = 1 + 2 + 3 + 4 + 5 + \dots + n = \frac{n(n+1)}{2} \text{ and}$$

$$\sum (x_i)^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}, \text{ respectively;}$$

show that their mean and variance is  $\frac{n+1}{2}$  and  $\frac{n^2-1}{12}$ , respectively.

#### Answer

$$1. \quad \bar{x} = \frac{1}{n} \sum x_i = \frac{1}{n}(1 + 2 + 3 + 4 + 5 + \dots + n) = \frac{1}{n} \left( \frac{n(n+1)}{2} \right) = \frac{n+1}{2}$$

$$\text{Variance } (\sigma^2) = \frac{1}{n} \sum x_i^2 - (\bar{x})^2 = \frac{1}{n} \left( \frac{n(n+1)(2n+1)}{6} \right) - \left( \frac{n+1}{2} \right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= (n+1) \left[ \frac{2n+1}{6} - \frac{n+1}{4} \right] = (n+1) \left[ \frac{4n+2-3n-3}{12} \right] = (n+1) \left[ \frac{n-1}{12} \right]$$

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

### Assessment criteria

Extend understanding, analysis and interpretation of data arising from problems and questions in daily life to include standard deviation.

### Answers to Tasks of Unit 14 in the Student's Book

#### Task 14.1 on page 281

1.
  - (a)  $\bar{x} = 5$  and  $\sigma = 2$
  - (b)  $\bar{x} = 8.5$  and  $\sigma = 1.80$
  - (c)  $\bar{x} = 18.8$  and  $\sigma = 6.46$
2.
  - (a)  $\bar{x} = 10.834$  and  $\sigma = 4.10$
  - (b)  $\bar{x} = 3.42$  and  $\sigma = 1.91$
  - (c)  $\bar{x} = 205$  and  $\sigma = 3.16$

**Task 14.2 on page 283**

1. The inter-quartile range is 30
2. The standard deviation is 14.3614
3. The variance is 8. The standard deviation is  $2\sqrt{2}$
4.
  - a)  $x = 11$
  - b) The modes are 4, 10 and 11
  - c) The median is 10
  - d) The range is 16
  - e) The mean deviation is  $\frac{1}{n} \sum |x_1 - \bar{x}| = \frac{1}{10} (40) = 4$
  - f) The standard deviation is 4.8785
  - g) 9.5
  - h) 75

# Topic area: Statistics and probability

## Sub-topic area: Combinatorial analysis and probability

# UNIT 15 Combinatorics

**Number of periods: 18**

### Learning objectives

By the end of this unit, the student must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define the combinatorial analysis</li><li>• Recognise whether repetition is allowed or not, and if order matters or not in performing a given experiment</li><li>• Construct Pascal's triangle</li><li>• Distinguish between permutations and combinations</li></ul>	<ul style="list-style-type: none"><li>• Determine the number of permutations and combinations of "n" items, "r" taken at a time.</li><li>• Use counting techniques to solve related problems.</li><li>• Use properties of combinations</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the importance of counting techniques</li><li>• Show concern on how to use the counting techniques</li></ul>

### Content

1. Counting techniques:
  - Venn diagram
  - Tree diagrams
  - Contingency table
  - Multiplication principles

2. Arrangements and permutations:
  - Arrangements with or without repetition
  - Permutations with or without repetition
3. Combinations:
  - Definitions and properties
  - Pascal's triangles
  - Binomial expansion

## Materials required

Manila papers, graph papers, ruler, digital components including calculators.

## Generic competences

- Cooperation
- Communication
- Problem-solving
- Research

## Cross-cutting issues

- **Inclusive education**

People with special needs must be included in solving the country problems.

- **Gender education**

Group work to consist of both gender in full participation. The examples given also have both gender in the groups.

## Teaching and learning activities

### Introductory activity

Explain to learners the meaning of combinatorics. Revise with them Venn diagrams – to include complement of a set, intersections and unions of sets. Explain what disjoint sets are. Let them tackle Task 15.1 on page 285 of the Student's Book.

### Main activities

1. Discuss with learners what tree diagrams are and how they are used in representing relationships.
2. Discuss with the students and explain to them the use of a contingency table.

3. Give them the Mental task on page 288 of the Student's Book to introduce them to the generalities of permutations and combinations.
4. Explain the factorial notation.
5. Give them Activity 15.1 on page 288 of the Student's Book to introduce them to the real concept of permutations. After that, guide them to be able to distinguish between those with repetition and those without.
6. Discuss with them the arrangement in a circle formation as brought out on page 291 of the Student's Book.
7. Explain and give examples and practice questions on conditional arrangements. Let them do Task 15.2 on page 293 of the Student's Book.
8. Give them Activity 15.2 on page 294 of the Student's Book to assist them grasp the concept of combinations. Expound on the definitions and explain using examples the different possible scenarios on combinations.
9. Let them do Task 15.3 on page 296 of the Student's Book.
10. Explain the concept of Pascal's triangles and guide students in linking it to the binomial expansion. Let them do Task 15.4 of page 299 and Task 15.5 of page 301 in groups or pairs.

## **Reinforcement activities**

Use combinations and permutations to determine the number of ways a random experiment occurs.

## **Learners of varying strengths and abilities**

### **(a) Physical impairment**

- Use cooperative learning, for instance through group work and discussions. Those having difficulty with manipulative tasks should be assisted by the others in the group.
- Mix students with special needs with the rest so as to be helped
- Provide these students with frequent progress checks.

### **(b) Hearing impairment**

- Record portions of textbooks, trade books, and other printed materials in audio so students can listen (with earphones) to an oral presentation of necessary material.
- Providing written or pictorial directions to those with hearing problems

- Using of concrete objects such as models, diagrams, samples, and the like to those with so as to demonstrate what you are saying by using touchable items. They can also rewrite content such as charts on large manila paper in their group work
- Facing the learner while you speak might help learners with a hearing impairment
- Use large writing on the blackboard and on visual aids

**(c) Intellectual impairment**

- Try to understand the specific talents of the learner and develop them;
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

## Additional tasks

### Task 15A - For slow learners

1. If  $(n + 2)! = 20n!$ , find  $n$ .
2. (a) If license plates of cars in Rwanda are made of the letter R followed by any two letters of the alphabet and any three digits followed by any one letter, how many possible different license plates can be made?  
  
(b) If any MTN Rwanda line must contain 10 digits where the first three of which are 078, find the number of all possible different MTN lines.

#### Answers

1.  $(n + 2)(n + 1)n! = 20n!$   
 $(n + 2)(n + 1) = 20$   
 $n^2 + 3n + 2 - 20 = 0$   
 $n^2 + 3n - 18 = 0$   
 $(n - 3)(n + 6) = 0$   
 $n = 3$  and  $n = -6$  must be rejected  
 Thus,  $n = 3$  is the only solution

2. (a) We have 26 alphabet letters and 10 digits to choose from where the first letter must be R and there is one possible choice of R.

The number of possible different license plates is

$$1 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 26 = 17,576,000$$

- (b) We have 10 digits to choose from where the first three digits must be 078

The number of all possible different MTN lines is

$$1 \times 10 = 10,000,000$$

### Task 15B - For talented learners

1. Calculate the number of different ways in which six people can form
- (a) A queue (that is, a single file) of six people
  - (b) A queue of two people and another queue of four people
  - (c) a group of two people and another group of four people
  - (d) two groups of three people each
  - (e) three pairs
  - (f) first, second and third pairs
2. How many odd numbers between 2,000 and 3,000 can be formed from the digits 1, 2, 3, 4, 5 and 6 using each of them only once in each number?

#### Answers

1. (a) The number of different ways in which six people can form a queue is  $6! = 720$  ways
- (b) This is a permutation of 6 people in 6 places. The number of ways is  $6! = 720$
- (c) Here we find the ways by forming one group of 2 people from 6 and the other group will be formed automatically. The number of ways is  $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2 \times 4!} = 15$
- (d) Here we find the ways by forming one group of 3 people from 6 and the other group will be formed automatically. The number of ways is  $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \frac{6 \times 5 \times 4}{6} = 20$
- (e) This is to form the group of two people from six.

The number of ways is  $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2 \times 4!} = 15$

(f) The number of ways we can form any of the three groups is

$$\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2 \times 4!} = 15.$$

The number of different permutations is  $3! = 6$

Therefore, the number of different ways of having the first, second and third pairs is  $15 \cdot 6 = 90$ .

2. Because the number is between 2000 and 3000, it has only four digits and the first must only be 2. The number is odd. So, the last digits must be taken from 1, 3 and 5. Once the first and the last digits are placed there are four digits available for second place and three for the third place.

	1st	2nd	3rd	4th (Last)
Number of ways of selecting digit	1	4	3	3

There are  $1 \cdot 4 \cdot 3 \cdot 3 = 36$  numbers.

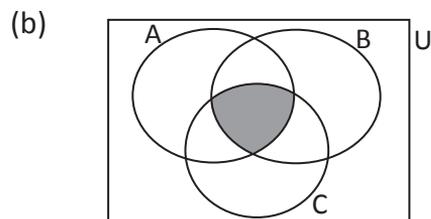
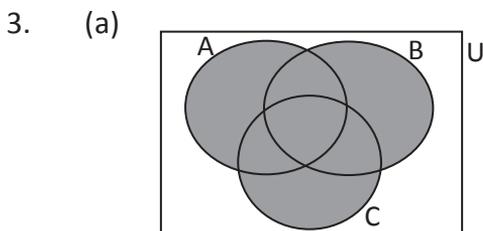
## Assessment criteria

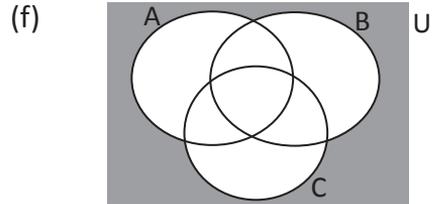
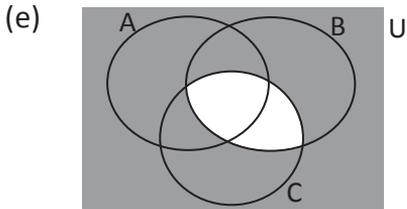
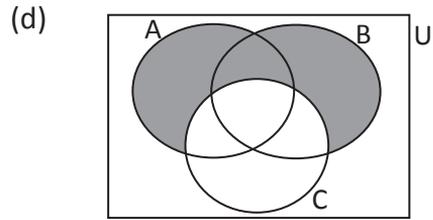
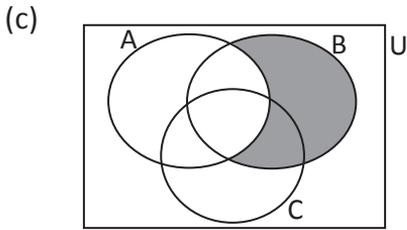
Calculate accurately combinations or permutations of 'n' items, taking 'r' at a time.

## Answers to Tasks of Unit 15 in the Student's Book

### Task 15.1 on page 285

1. (a) {1, 3, 4, 5, 6, 7}                      (b) {3, 5, 7}  
 (c) {4, 6}                                      (d) {1, 2, 3, 5, 7}  
 (e) {1}    (f) {2}
2. (a) {3, 4, 5, 6}                              (b) {1, 4, 9, 16, 25}  
 (c) {2, 3, 4, 5, 6}                            (d) {1, 11, 13, 143}  
 (e) {1}    (f) {1}





**Task 15.2 on page 293**

1. (a) 24 (b) 120
2. 720
3. 3,628,800
4. (a) 125 (b) 60
5. 120
6. 5,040
7. 168
8. (a) 240 (b) 600
9. (a) 1,000 (b) 720 (c) 990
10. (a) 103,680 (b) 34,560

**Task 15.3 on page 296**

1. (a) 6 (b) 84 (c) 1820
2. 1140
3. 525
4. (a) 462 (b) 56 (c) 20
5. 1,260
6. (a) 924 (b) 34,650
7. (a) 729 (b) 28
8. 10

9. 61 (including the given straight line)  
 10. 56  
 11. (a) 286 (b) 84

**Task 15.4 on page 299**

1.  $x^3 + 9x^2 + 27x + 27$   
 2.  $x^4 - 8x^3 + 24x^2 - 32x + 16$   
 3.  $x^4 + 4x^3 + 6x^2 + 4x + 1$   
 4.  $8x^3 + 12x^2 + 6x + 1$   
 5.  $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$   
 6.  $p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$   
 7.  $8x^3 + 36x^2 + 54x + 27$   
 8.  $x^5 - 20x^4 + 160x^3 - 640x^2 + 1280x - 1,024$   
 9.  $81x^4 - 108x^3 + 54x^2 - 12x + 1$   
 10.  $1 + 20a + 150a^2 + 500a^3 + 625a^4$

**Task 15.5 on page 301**

1. (a)  $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$   
 (b)  $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$   
 (c)  $64 + 192p^2 + 240p^4 + 160p^6 + 60p^8 + 12p^{10} + p^{12}$   
 (d)  $32h^5 - 80h^4k + 80h^3k^2 - 40h^2k^3 + 10k^4 - k^5$   
 (e)  $x^3 + 3x + 3x^{-1} + x^{-3}$   
 (f)  $z^8 - 4z^6 + 7z^4 - 7z^2 + \frac{35}{8} - \frac{7}{4}z^{-2} + \frac{7}{16}z^{-4} - \frac{1}{16}z^{-6} + \frac{1}{256}z^{-8}$
2.  $64x^5 + 160x^{-1} + 20x^{-7}$   
 3. 0, 1 (trivial) and 6.  
 4. 2  
 5. 30.43168  
 6. (a) 560 (b) -590,625 (c) -720 (d) -448  
 (e) 1,966,080

# Topic area: Statistics and probability

## Sub-topic area: Combinatorial analysis and probability

UNIT

# 16 Elementary probability

**Number of periods: 7**

### Learning objectives

By the end of this unit, students must be able to do the following:

Knowledge and understanding	Skills	Attitudes and values
<ul style="list-style-type: none"><li>• Define probability and explain probability as a measure of chance</li><li>• Distinguish between mutually exclusive and non-exclusive events and compute their probabilities</li></ul>	<ul style="list-style-type: none"><li>• Use and apply properties of probability to calculate the number possible outcomes of occurring event under equally likely assumptions</li><li>• Determine and explain expectations from an experiment with possible outcomes</li></ul>	<ul style="list-style-type: none"><li>• Appreciate the use of probability as a measure of chance</li><li>• Show concern on patience, mutual respect, tolerance and curiosity in the determination of the number of possible outcomes of a random experiment</li></ul>

### Content

1. Concepts of probability:
  - Random experiment
  - Sample space
2. Finite probability spaces
3. Sum and product laws
4. Conditional probability

### Materials required

Manila paper, graph paper, ruler, digital components including calculators.

## Generic competences

- Problem solving
- Cooperation
- Communication
- Research

## Cross-cutting issues

- **Standardization culture**

Learners encouraged to adhere to rules and standards of calculations and drawing of graphs.

- **Gender education**

Groups consist of mixed gender and all are encouraged to participate. Examples given in the Student's Book and Teacher's Guide are inclusive of all gender.

## Teaching and learning activities

### Introductory activity

Use counting techniques and concepts of probability to determine the probability of possible outcomes of events occurring under equally likely assumptions.

### Main activities

1. Let learners discuss problems associated with gambling and report their results to the group. This is the requirement of Activity 16.1 on page 303 of the Student's Book.
2. Assist them to define probability and explain probability as a measure of chance and explain expectations from an experiment with possible outcomes.
3. Guide students to distinguish between mutually exclusive and non-exclusive events and compute their probabilities.
4. Help them on how to use and apply properties of probability to calculate the number of possible outcomes occurring under equally likely assumptions
5. Introduce permutations and combination in probability theory by giving them Activity 16.2 on page 308 of the Student's Book. Learners are given a task of sitting 3 men and 4 women at random in a row. In groups, they discuss about the probability that all the men are seated together then they give feedback.

6. Explain to them the concept, to include probability spaces. Let them tackle Task 16.1 found on page 310 of the Student's Book.
7. Explain sum and product laws and let them tackle, in pairs, Task 16.2 on page 313 of the Student's Book.
8. Take them through the various types of conditional probability and let them work on Task 16.3 on page 315 of the Student's Book.

## Reinforcement activities

Use counting techniques and concepts of probability to determine the probability of possible outcomes of events occurring under equally likely assumptions

## Learners of varying strengths and abilities

### (a) Gifted and talented

Guide the learners to study units ahead of others. You can provide more advanced material to the learner.

Give them individual exercises to work on as homework.

### (b) Slow learners

Give them extra tuition. Teach them to master the material skills that are commensurate with their ability levels before introducing complex components.

### (c) Intellectual impairment

- Try to understand the specific talents of the learner and develop them;
- Break the task down into small steps or learning objectives. Ensure learners start with what they can already do, then move on to a new more challenging task
- Give the learner lots of practice and time. It helps to ensure the child has mastered a skill.

## Additional tasks

### Task 16A - For slow learners

1. A fair (unbiased) coin is tossed 5 times. Find the probability of obtaining
  - (a) 5 tails
  - (b) Exactly 4 tails
  - (c) At least one head

## Answer

1. We are given an unbiased coin which is tossed 5 times.  $P(H) = P(T) = \frac{1}{2}$ .
- (a)  $P(\text{obtaining 5 tails}) = P(T, T, T, T, T)$   
 $= P(\text{obtaining no head}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$
- (b)  $P(\text{exactly 4 tails}) =$   
 $P(H, T, T, T, T) + P(T, H, T, T, T) + P(T, T, H, T, T) + P(T, T, T, H, T)$   
 $+ P(T, T, T, T, H) = 5\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{5}{32}$
- (c)  $P(\text{at least one head}) = 1 - P(\text{obtaining no head}) = 1 - \frac{1}{32} = \frac{32-1}{32} = \frac{31}{32}$

## Task 16B - For talented learners

1. Show that sets A and B are independent if and only if  $P(A \cap B) = P(A) \times P(B)$
2. Each seat of Gacaca Judiciary court of a cell is made up of 9 persons of integrity.
- (a) How many ways of choosing a committee of coordination of judiciary if each member of Gacaca seat is a candidate to one post and selections for the posts are made in 3 successive phases:  
Phase 1: one president  
Phase 2: first and second vice presidents  
Phase 3: two secretaries
- (b) If the seat of Gacaca Court of the cell is composed of 4 ladies and 5 men:
- what is the probability of choosing a lady as the president of the judiciary court?
  - what is the probability for the two posts of first and second vice presidents to be occupied by ladies?
  - what is the probability of choosing two secretaries of different sexes if the president and one of the vice presidents are men?

## Answers

1. We know that  $P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$  this gives us  $P(A \cap B) = P(A) \times P(B)$   
and  $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = P(B)$  this gives us  $P(A \cap B) = P(A) \times P(B)$
- 2.
- (a) We select 1 out of 9, then 2 out of 8 remaining (those can interchange positions), then 2 out of 6 remaining: The number of ways is

$$\binom{9}{1} \times \binom{8}{2} P_2^2 \times \binom{6}{2} = \frac{9!}{1! \times 8!} \times 2 \times \frac{8!}{2! \times 6!} \times \frac{6!}{2! \times 4!} = 9 \times 2 \times 28 \times 15 = 7560 \text{ or}$$

$$\binom{9}{1} \times P_2^8 \times \binom{6}{2} = \frac{9!}{1! \times 8!} \times \frac{8!}{6!} \times \frac{6!}{2! \times 4!} = 9 \times 56 \times 15 = 7560$$

(b) i) The probability of choosing a woman is  $\frac{4}{9}$ .

ii) Here is a case where we have 4 women if the elected president is a man and the case where we have 3 women if the elected president is a woman.

$$\text{The probability is } \frac{P_2^4 + P_2^3}{P_2^8} = \frac{\frac{4!}{2!} + \frac{3!}{1!}}{\frac{8!}{6!}} = \frac{12 + 6}{56} = \frac{18}{56} = \frac{9}{28}$$

Or since we are interested in that the two vice presidents are women, we can ignore the order and we find the probability

$$\text{is } \frac{\binom{4}{2} + \binom{3}{2}}{\binom{8}{2}} = \frac{6 + 3}{28} = \frac{9}{28}$$

iii) Here we remain by 3 men and 3 women and one on each gender is

selected.

$$\text{The probability is } \frac{\binom{3}{1} \times \binom{3}{1}}{\binom{6}{2}} = \frac{3 \times 3}{15} = \frac{9}{15} = \frac{3}{5}$$

## Assessment criteria

Use counting techniques and concept of probability to determine the probability of possible outcomes occurring under equally likely assumptions.

## Answers to Tasks of Unit 16 in the Student's Book

### Task 16.1 on page 310

- (a)  $\frac{1}{2}$                       (b)  $\frac{1}{2}$                       (c)  $\frac{1}{2}$
- (a)  $\frac{1}{4}$                       (b)  $\frac{3}{4}$                       (c)  $\frac{7}{24}$

(d)  $\frac{11}{12}$
- (a)  $\frac{1}{17}$                       (b)  $\frac{15}{34}$
- (a)  $\frac{1}{6}$                       (b)  $\frac{5}{126}$

### Task 16.2 on page 313

- $P(A) = \frac{3}{8}$ ,  $P(B) = \frac{5}{12}$  and  $P(A \cap B) = \frac{1}{4}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{8} + \frac{5}{12} - \frac{1}{4} = \frac{9+10-6}{24} = \frac{13}{24}$$

2.  $P(A - B) = 0.3$ ,  $P(B - A) = 0.4$ ,  $P(A' \cap B') = 0.1$

(a) We know that  $P(A' \cap B') = P(A \cup B)' = 0.1$

$$P(A \cap B) = 1 - [P(A - B) + P(B - A) + P(A \cup B)']$$

$$= 1 - [0.3 + 0.4 + 0.1] = 1 - 0.8 = 0.2$$

(b)  $P(A) = P(A - B) + P(A \cap B) = 0.3 + 0.2 = 0.5$

(c)  $P(A) = P(B - A) + P(A \cap B) = 0.4 + 0.2 = 0.6$

3. A and B are independent events such that  $P(A) = \frac{3}{4}$ ,  $P(B) = \frac{5}{6}$ .

(a)  $P(A \cap B) = P(A) \times P(B) = \frac{3}{4} \times \frac{5}{6} = \frac{15}{24} = \frac{5}{8}$

(b)  $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{4} + \frac{5}{6} - \frac{5}{8} = \frac{18+20-15}{24} = \frac{23}{24}$

(c)  $P(A \cap B') = P(A) \times P(B') = \frac{3}{4} \times \frac{1}{6} = \frac{3}{24} = \frac{1}{8}$

(d)  $P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B) = 1 - \frac{5}{8} = \frac{8-5}{8} = \frac{3}{8}$

4. We are given  $P(6) = \frac{1}{3}$

(a)  $P(2 \text{ sixes}) = P(6,6) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

(b)  $P(\text{at least one six}) = P(6,6) + P(6,\bar{6}) + P(\bar{6},6) = \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3}$   
 $= \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{5}{9}$

5. An unbiased die is thrown three times

(a)  $P(\text{obtaining three sixes}) = P(6, 6, 6) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

(b)  $P(\text{obtaining exactly two sixes}) = P(6, 6, \bar{6}) + P(6, \bar{6}, 6) + P(\bar{6}, 6, 6)$

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$$

$$= \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216} = \frac{5}{72}$$

(c)  $P(\text{obtaining at least one six})$

$$= P(6, 6, 6) + P(6, 6, \bar{6}) + P(6, \bar{6}, 6) + P(\bar{6}, 6, 6) + P(\bar{6}, \bar{6}, 6) + P(\bar{6}, 6, \bar{6})$$

$$+ P(6, \bar{6}, \bar{6})$$

$$= \frac{1}{216} + 3\left(\frac{15}{216}\right) + 3\left(\frac{25}{216}\right) = \frac{1+45+75}{216} = \frac{121}{216}$$

### Task 16.3 on page 315

1. The sample space is  $\Omega = \{(3,3), (4,2), (2,4), (5,1), (1,5)\}$

$$P(\text{one is 2}) = \frac{2}{5}$$

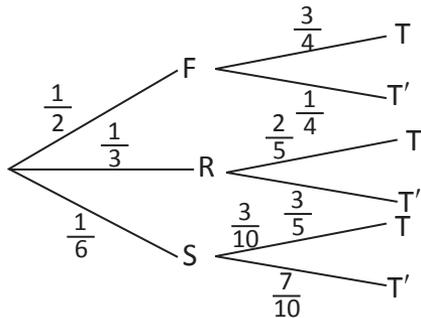
2. (a)  $P(3 \text{ red marbles are followed by } 1 \text{ blue marble, marble is replaced})$   
 $= \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8} \times \frac{3}{8} = \frac{375}{4096}$

(b)  $P(3 \text{ red marbles are followed by } 1 \text{ blue marble, marble is not replaced.})$

$$= \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} = \frac{180}{1680} = \frac{3}{28}$$

3. We are given  $P(\text{fine day}) = \frac{1}{2}$ ,  $P(\text{raining day}) = \frac{1}{3}$ ,  $P(\text{snowing day}) = \frac{1}{6}$ .

We make use of a tree diagram below.



$$P(\text{Student is on time} | \text{fine day}) = P(T|F) = \frac{3}{4}$$

$$P(\text{student is on time} | \text{raining day}) = P(T|R) = \frac{2}{5}$$

$$P(\text{Student is on time} | \text{snowing day}) = P(T|S) = \frac{3}{10}$$

(a)  $P(\text{Student is on time on any given day})$

$$= P(T) = P(T|F) + P(T|R) + P(T|S)$$

$$= \frac{1}{2} \times \frac{3}{4} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{6} \times \frac{3}{10} = \frac{3}{8} + \frac{2}{15} + \frac{3}{60}$$

$$= \frac{45 + 16 + 6}{120} = \frac{67}{120}$$

(b)  $P(R|T') = \frac{P(R \cap T')}{P(T')} = \frac{\frac{1}{3} \times \frac{3}{5}}{\frac{67}{120}} = \frac{\frac{3}{15}}{\frac{67}{120}} = \frac{3}{15} \times \frac{120}{67} = \frac{24}{67}$

4. (a) We are given a tin with 4 red and 6 blue marbles. Three marbles are withdrawn without replacement

(a)  $P(\text{first two are red and the third is blue}) = \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} = \frac{72}{720} = \frac{1}{10}$

(b)  $P(\text{two are red marbles and one is blue})$

$$= \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8}$$

$$= 3 \left( \frac{72}{720} \right) = \frac{3}{10}$$

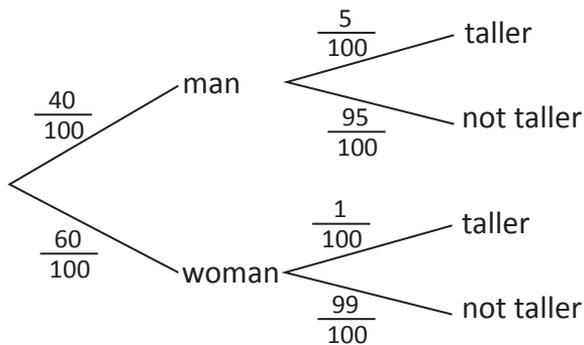
5. The probability that the fifth card dealt to him is the fourth ace is  $\frac{1}{48}$ .

6. We are given that  $P(\text{man is taller than 180 cm}) = \frac{5}{100}$

$$P(\text{woman is taller than 180 cm}) = \frac{1}{100}$$

$$P(\text{Student is a man}) = \frac{40}{100} \text{ and } P(\text{Student is a woman}) = \frac{60}{100}$$

We make use of the tree diagram below:



$$P(\text{Student is taller than 180 cm}) = \frac{40}{100} \times \frac{5}{100} + \frac{60}{100} \times \frac{1}{100} = \frac{10}{500} + \frac{3}{500} = \frac{13}{500}$$

$P(\text{Student is a woman} | \text{is taller than 180 cm})$

$$= \frac{P(\text{Student is a woman and is taller than 180 cm})}{P(\text{Student is taller than 180 cm})}$$

$$= \frac{\frac{60}{100} \times \frac{1}{100}}{\frac{13}{500}} = \frac{\frac{60}{10000}}{\frac{13}{500}} = \frac{3}{500} \times \frac{500}{13} = \frac{3}{13}$$

If a student is selected at random and found to be taller than 180 cm, then the probability that this student is a woman is  $\frac{3}{13}$ .

## Answers to Practice Tasks

### Practice Task I on page 317

1.

p	q	$\sim p$	$\sim q$	$p \vee q$	$(p \vee q)$	$(\sim p) \wedge (\sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

2. (a)  $\exists x \in \mathbb{R} : x^2 \leq 0$

(b)  $\exists x \in \mathbb{R}; \forall n \in \mathbb{N} : x \geq n$

3. (a)  $1, 105^\circ$  (b)  $1, 30^\circ$  (c)  $1, 300^\circ$   
 (d)  $1, 330^\circ$

4. (a)  $x = -\frac{3}{7}$  (b)  $a = 4$  (c)  $a = \pm 64$

5. (a)  $p \geq 9$  and  $p \leq 1$  (b)  $p \geq 5$  and  $p \leq 1$

6. (a)  $\frac{1}{3}$  (b)  $-\frac{1}{4}$  (c)  $\sqrt[3]{\frac{2}{9}}$   
 (d)  $-\frac{1}{4}$  (e) 5

7. Vertical asymptotes V.A  $\equiv x = -2$  and V.A  $\equiv x = 2$

Horizontal asymptote H.A.  $\equiv y = 0$

8.  $\left[\frac{1}{12}, \frac{1}{4}\right]$

9. Set of solutions is  $\{(1, 1); (\frac{1}{4}, 4)\}$

10. (a)  $\frac{3}{5}$  (b)  $-\frac{4}{5}$  (c)  $-\frac{3}{5}$

11.  $\frac{dg}{dt} = \frac{1}{2\sqrt{t}}$

12.  $0 < a < \frac{4}{9}$

13. 1,760 ways.

14.  $P(x) = x^2 - x + 3$

15. Number of permutations is  $\frac{8!}{3!2!} = \frac{40,320}{6 \times 2} = 3,360$

16. (a)  $\text{Dom } f = (-\infty, 3) \cup (3, +\infty)$

(b) x-intercept is  $(-1, 0)$ . y-intercept is  $(0, \frac{1}{9})$

- (c) The vertical asymptote is  $VA \equiv x = 3$ . The horizontal asymptote is  $HA \equiv y = 0$
- (d)  $f'(x) = \frac{(x+5)}{(x-3)^3}$  and  $f''(x) = \frac{2(x+9)}{(x-3)^4}$
- (e) The graph of  $f$  has a local minimum point at  $(-5, -\frac{1}{16})$
- (f) The graph of  $f$  is increasing on  $(-5, 3)$ ,  $f$  is decreasing on  $(-\infty, 5)$  and  $(3, +\infty)$ .
- (g) The graph of  $f$  has inflection point at  $(-9, -\frac{1}{18})$ .
- (h) The graph of  $f$  is concave up on  $(-9, 3)$  and  $(3, +\infty)$ , and concave down on  $(-\infty, -9)$ .
17. (a)  $f(x) \geq -3$                       (b)  $f(x) \geq -5$                       (c)  $f(x) \geq 0$
- (d)  $0 \leq f(x) \leq \frac{1}{2}$
18. (a)  $P(2 \text{ survive}) = 0.384$                       (b)  $P(3 \text{ survive}) = 0.512$
19. (a) The mean is 2.4                      (b) The median is 2
- (c) The mode is 3                      (d) The sample standard deviation is 1.66
20. (a)  $\tan 2\theta = \frac{24}{7}$                       (b) The height of the flagpole is 10m

### Practice Task 2 on page 320

1. (a) The centre is at  $(-4, 1)$  and radius is 4 units of length
- (b) The centre is at  $(-\frac{1}{2}, -\frac{3}{2})$  and radius is  $\frac{3\sqrt{2}}{2}$  units of length
- (c) The centre is at  $(-3, 0)$  and radius is  $\sqrt{14}$  units of length
2. (a) The probability that dictionary is selected is  $\frac{1}{3}$
- (b) The probability that 2 novels and 1 book of poems are selected is  $\frac{5}{14}$
3.  $A^{-1} = 5I + A$
4. (a)  $-4 < x < -\frac{2}{3}$                       (b)  $2 < x < 3$
- (c)  $x < -2$
5. The  $*$  operation admits  $e = -3$  as the identity element in  $\mathbb{Z}$ .
- The  $*$  operation admits  $x' = -x - 6$  as a symmetric (inverse) of  $x$  in  $\mathbb{Z}$ .
6. (a) 0                      (b)  $\frac{1}{2}$                       (c)  $\frac{1}{4}(\sqrt{6} - \sqrt{2})$
- (d)  $-(2 + \sqrt{3})$                       (e)  $\frac{1}{4}(\sqrt{6} - \sqrt{2})$                       (f)  $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

7. (a)  $\sin 3\theta$  (b) 0
8. (a)  $A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$  (b)  $B^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$
- (c)  $(AB)^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$  (d)  $(BA)^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$
- (e)  $A^{-1}B^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$  (f)  $B^{-1}A^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$
9. (a)  $\frac{\log 3}{2 \log 3 - 3 \log 2} = 9.33$   
 (b) The answer is  $-\frac{1}{3} < x < 3 \Rightarrow x \in ]-\frac{1}{3}, 3[$
10. (a) The amount of material in the original sample is 80 g.  
 (b) The half-life is 100 years.  
 (c) It will take 632 years for the material to decay to 1.
11. The vertex is  $(-\frac{3}{4}, -\frac{49}{8})$ , and the axis is  $X = -\frac{3}{4}$
12. (a) For two distinct real roots  $\Delta > 0$  and so  $k < -3$  or  $k > 1$ .  
 (b) For no real roots  $\Delta < 0$  and so  $-3 < k < 1$ .
13. (a)  $(g \circ f)(x) = g[f(x)] = g(x + 2) = 2(x + 2) + 3 = 2x + 4 + 3 = 2x + 7$   
 (b)  $(f \circ g)(x) = f[g(x)] = f(2x + 3) = (2x + 3) + 2 = 2x + 3 + 2 = 2x + 5$   
 (c)  $(f \circ f)(x) = f[f(x)] = f(x + 2) = (x + 2) + 2 = x + 4$   
 (d)  $(g \circ g)(x) = g[g(x)] = g(2x + 3) = 2(2x + 3) + 3 = 4x + 6 + 3 = 4x + 9$
14. The three vectors are linearly dependent.
15. (a) For any vector  $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$  in  $\mathbb{R}^2$ , we have  $c_1 \frac{5x-2y}{7}$  and  $c_2 = \frac{3y-4x}{7}$  for  $\vec{u} = c_1 \vec{v} + c_2 \vec{w}$ .  
 (b)  $c_1 \vec{v} + c_2 \vec{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  implies  $c_1 = c_2 = 0$ .
16. (a) (i)  $\frac{1}{52}$  (ii)  $\frac{7}{26}$  (iii)  $\frac{10}{13}$   
 (b)  $\frac{13}{51}$
17. The perimeter of the triangle is 156 cm.

18. (a)  $(f \circ g)(x) = (2 - x)^2$  and  $\text{Dom}(f \circ g) = \mathbb{R}$  and  $\text{range}(f \circ g) = [0, +\infty)$   
 (b)  $(g \circ f)(x) = 2 - x^2$  and  $\text{Dom}(g \circ f) = \mathbb{R}$  and  $\text{range}(g \circ f) = [-\infty, 2)$
19. (a)  $a = 1$  and  $b = -5$   
 (b)  $f(x) = (x - 1)(x - 1)(x + 3)$
20. (a) 5,040      b) 720

### Practice Test 3 on page 322

1. The distance is  $\frac{3}{5}$  unit of length.
2. (a)  $\frac{7}{20}$       (b)  $\frac{11}{20}$       (c)  $\frac{3}{20}$   
 (d)  $\frac{3}{4}$
3.  $T \equiv y = 4x - 20$  or  $T \equiv y = -4x + 4$  and the tangents meet at  $(3, -8)$
4. The number of arrangements in the word **MIYOVE** starting with a consonant is  $3(5!) = 3 \times 5 \times 4 \times 3 \times 2 \times 1 = 360$
5. (a) The centre of the circle is  $(-2, -3)$  and the radius is 4.  
 (b) The shortest distance between the centre of the circle and the line L is 4.  
 (c) Since the shortest distance between the centre of the circle and line L is exactly equal to the radius of the circle, then the line L is the tangent to the circle.  
 (d) The point of tangency is  $(-\frac{22}{5}, \frac{1}{5})$
6. They can be formed into 36 numbers.
7. (a) 1      (b)  $\frac{7}{8}$       (c) 2      (d) -2  
 (e)  $-\frac{1}{2}$
8. (a) 120 permutations      (b) 240 permutations
9. (a)  $6561 - 34992x + 81648x^2 - \dots$       (b)  $1 - 5x + \frac{45}{4}x^2 + \dots$
10. (a) The slope at  $x = 2$  is 72  
 (b) Tangent has equation  $T \equiv y = -1$   
 (c) The points are  $(0, 2)$ ,  $(-1, -1)$ , and  $(1, -1)$
11. The graph of the function  $f$  is increasing on the interval  $[3, \infty)$ .  
 The graph of the function  $f$  is decreasing on the interval  $(-\infty, 3]$ .

12. (a) The mean is 6. (b) The median is 5  
(c) The mode does not exist. (d) The range is 9  
(e) The variance is 10.  
(f) The standard deviation is  $\sqrt{10}$ .
13. (a) The inflection numbers of  $f$  are 0 and 2 and the inflection points are (0,5) and (2, -11).  
(b) The graph of  $f$  is bending up on the interval  $(-\infty, 0)$  and  $(2, \infty)$  and bending down on the interval  $(0, 2)$ .
14.  $k = -\frac{1}{6}$
15. (a) The slope is 0 at the points where  $x = -1 + \sqrt{2}$  and  $x = -1 - \sqrt{2}$ .  
(b) The slope is 2 at the points where  $x = -3$  and  $x = 1$ .  
(c) The slope is -1 at the point where  $x = 0$  and  $x = -2$ .
16. The height of the tree is 50 m.
17. The largest possible value for  $xy$  is 100.
18. The angle of elevation of the sun is  $40^\circ$ .
19. (a)  $\frac{1}{9}$  (b)  $\frac{1}{35}$
20. (a)  $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$   
(b)  $x = 1, y = 2$

## REFERENCES

1. Boutriau-Philippe, E. et al, ***Savoir Faire en Mathématique***, 2e Année, 8e edition, De Boeck-Wesmael, Bruxelles,1994
2. Delord, R., ***Mathématiques, 4e année***, Hachette,1992
3. Rich B & Thomas C. (2009). ***Schaum's Outline, Geometry***
4. Rwanda Education Board (2015). ***Advanced Level Mathematics Syllabus for Science Combinations***