Advanced Mathematics

for Rwanda Secondary Schools

Teacher's Guide

Senior Five

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Section 1: General introduction

1.1. General introduction to the new curriculum

The curriculum for Rwandan schools at primary and secondary levels has been changed from knowledge and content bases to competence based (CBC). CBC is of great importance in aligning Rwanda's education to the social and economic demands of society. It also presents answers to concerns about the capability and employability of school graduates.

1.2. General guidance to teachers

In Advanced mathematics-Learner's Book Five, there are 10 units. There are many activities to be done by learners before a new lesson. This will help students to understand well the lesson.

Teacher must help students to do those activities. Form groups of at least six students and let them do the activities in those groups.

At the end of the lesson, there is a series of exercises which summarise the lesson taught. As teacher, let the students do those exercises and correct them on chalkboard.

Also, at the end of each unit, there is a series of exercises which summarise the whole unit.

1.3. List of equipment needed for the subject

Students will need geometric instruments for sketching curves and scientific calculators for some calculations.

1.4. General guidance on assessment both formative and summative

Assessment is the use of a variety of procedures to collect information about learning and instruction. Formative assessment is commonly referred to as assessment for learning, in which focus is on monitoring student response to and progress with instruction. Formative assessment provides immediate feedback to both the teacher and student regarding the learning process. Formative and summative assessments contribute in different ways to the larger goals of the assessment process.

Teacher must provide oral or written feedback to students' discussion or work. For example, a teacher responds orally to a question asked in class; provides a written comment in a response or reflective journal; or provides feedback on student work.

At the end of each lesson, teacher must give to the students a small evaluation to see if they understood the lesson. Also at the end of the units, teacher must give the general test which summarises the whole unit. When assessments reflect the stated learning objectives, a well-designed end of unit test provides teachers with information about individual students (identifying any student who failed to meet objectives), as well as provides an overall indication of classroom instruction.

Although formative and summative assessments serve different purposes, they should be used ultimately within an integrated system of assessment, curriculum, and instruction. To be effective in informing the learning process, assessments must be directly integrated with theories about the content, instruction, and the learning process and must be valid and reliable for the purposes for which they are used.

1.5. Guidance on grading and reporting

Academic achievement may be measured in a variety of ways, including compositions, presentations, oral discussion, student work samples, observations, tests, and the products of project-based learning activities. Teachers should use the most current summative assessment data when determining achievement marks for the progress report. When determining what marks to use on daily and weekly assignments, remember that these marks should not conflict with the grades on the progress report. Teachers should not use letter grades when marking papers.

Some options may include:

- Raw scores or ratios (11/12 correct)
- Written feedback
- Rubric scores (if using 4, 3, 2, or 1 on papers, there should be guidance as to what these marks mean).

Keep in mind that work that is sent home provides parents with a general impression of how students are achieving in school but does not provide a complete picture. Other assessment data are collected that encompasses the progress report grade and some of these assessments are not sent home. Communication regarding progress should be ongoing.

Homework can be considered as part of the effort grade, but would not be used to grade academic achievement in elementary school since the function of homework is to provide practice in skill areas.

Achievement marks will be reported on a 4-point scale and cannot be equated to former guidelines for letter grades. A grade of "4" indicates a high level of achievement; it communicates that a student has a strong understanding of all the concepts and skills taught for that standard during the quarter and can demonstrate understanding independently and with very few errors. When determining grades for students, teachers should consider the most current assessment data as evidence of learning. Earlier assessments may no longer be relevant if students have demonstrated further progress.

Section 2: Effective tips for brighter learners

Teacher must encourage learners to;

1. explain to each other as best they can the lesson learned.
2. have an exercise book, homework book and note book of course.
3. have geometric instruments and scientific calculator.
4. work out exercises and homework and check their answers to gain practice with every lesson.
5. do their own research on learned lessons or lessons to be learned next time.

Section 3: Effective tips for slow learners

Mathematics may be challenging for a slow learner, but not impossible. Slow learners also want to learn mathematics, but lack of learning ability and mathematics education system, they are not able to learn faster. The following are some techniques which can help slow learners:

1. Slow learners need more time to understand any problem or to find out the answer. Give extra time to slow learners. This will increase their confidence. Do not pressurize learners to perform on time beyond their ability. This will only decrease confidence.
2. Slow learners need extra attention. With a small students group, you can effectively respond to each student.
3. Environment is more potent than willpower. Create a fun environment for learners. Use new learning techniques, especially for slow learners. Teacher can provide mathematics games and activities to learners.

4. Build a helpful environment for learners. Encourage learners to ask questions and let them feel free to ask for any help.
5. Most slow learners face difficulty to understand the new concepts. Try to relate the new concepts with previous concepts. This will help them to catch the new concepts relatively fast.
6. One of the best ways to teach mathematics not only to slow learners but even for normal students is, explain concepts using real life examples.
7. Whenever possible, provide opportunities to show them their work. Let the students teach you about mathematics. This will help students to reduce mathematics fear.
8. Because slow learners need more time to understand the concepts, frequent reviewing can help them out. Reviewing mathematics concepts time to time will allow them to master the mathematics concepts.
9. Slow learners tend to have lack of confidence, if you pressurize them for time management or anything, this will only reduce their confidence.
10. Slow learners tend to have low confidence. Low confidence impedes anyone’s learning ability. If you reward them time to time, this will help them to raise their confidence.

Section 4: Extension knowledge and ideas for teachers

The following are the most important principles in mathematics teaching.

1.1. Principle 1: Let It Make Sense

Let us strive to teach for understanding of mathematical concepts and procedures, the **“why”** something works, and not only the “how”.

The **“how”** something works is often called **procedural understanding**: the learner knows how to work/solve a linear equation. It is often possible to learn the “how” mechanically without understanding why something works. Procedures learned this way are often forgotten very easily.

The relationship between the “how” and the “why” - or between procedures and concepts - is complex. **One doesn’t always come totally before the other**, and it also varies from learner to learner.

Try alternating the instruction: teach how to solve a linear equation, and let the learner practice. Then explain why it works.

As a teacher, don’t totally leave a topic until the student both knows “how” and understands the “why”.

Teacher can often test a student’s understanding of a topic by asking him “Tell me an example of where linear equation is used in daily life.”

1.2. Principle 2: Remember the Goals

Teacher must;

- cover the curriculum by the end of school year.
- make sure the learners have a lot formative and summative assessments.

Generally, teacher must

- enable the learners to understand information around us,
- prepare learners for further studies in mathematics,
- let learners see some beauty of mathematics and learn them to like it.

1.3. Principle 3: Know Tools

First of all of course comes a black or white board or paper — something to write on, then we have pencils, compass, protractor, ruler, eraser.... and the book the teacher is using.

Then we have computer software, interactive activities, animated lessons and such. There are workbooks, fun books, work texts, books, and online tutorials.

Teacher has to start somewhere, probably with the basics, and then add to his/ her "toolbox" little by little as you have opportunity. It's important to **learn how to use any tool** that teacher might acquire.

Basic tools

1. The board and/or paper to write on. Essential. Easy to use.
2. The learners' book and teacher's guide

The extras

1. Computer and projector
2. Internet connection

If computer lab is available at the school, teacher can show the learners how ICT is used in mathematics. For example;

- Writing mathematical expression using Microsoft Office, Word or other software tools like Math-Type,
- Sketching a function in Cartesian plane using Microsoft Office Excel or other software like Geogebra, Graph, Grapes, …
- Determine the mean, standard deviation, variance,… of a set of data using Microsoft Office Excel formulae or other software like Geogebra, …
- Performing mathematical operations using Microsoft Excel or other software like Mathlab, Geogebra, …

1.4. Principle 4: Living and Loving Mathematics

Mathematics teacher has to ensure that he/she

1. uses mathematics often in daily life,
2. likes mathematics,
3. loves mathematics,
4. is happy to teach mathematics.

Some ideas for teacher:

- Let it make sense. This alone can usually make quite a difference and learners will stay interested.
- Read through some fun math books. Get to know some interesting mathematics topics besides just schoolbook arithmetic. There are lots of story books (mathematics readers) that teach mathematics concepts.
- Consider including some mathematics history if you have the time.
- When you use mathematics in your daily life, explain how you're doing it, and include the children if possible. Figure it out together.

Section 5: Content map

The following table summarises every unit in Learners' Book.

	Unit 1 **Trigonometric formulae, equations and inequalities**	**Unit 2** **Sequence**
Number of Periods	35+ homework	25+ homework
Introduction	The techniques in trigonometry are used for finding relevance in navigation particularly satellite systems and astronomy, naval and aviation industries, oceanography, land surveying and in cartography (creation of maps). Now those are the scientific applications of the concepts in trigonometry, but most of the maths we study would seem (on the surface) to have little real-life application. Trigonometry is really relevant in our day to day activities.	By a sequence, we mean ordered list having a first element but not the last. That means, there is an order in the numbers; that is we actually have the first number, the second number and so on… A sequence is a set of real numbers with natural order.
Classroom Organisation	Whole class orientation; then working in pairs or in groups.	Whole class orientation; then working in pairs or in groups.
Equipment Required	• Instruments of geometry • Scientific calculator	Scientific calculator
Activities	Group work, pairing, practical, research	Groups discussion, research
Competences Practiced	• Team work • Creativity • Research	• Critical thinking • Team work • Analysis
Language Practice	Discussion in group, presentation of findings	Discussion in groups, presentation of findings
Vocabulary Acquisition		Sequence or progression, arithmetic and geometric sequence

	Unit 1 **Trigonometric formulae, equations and inequalities**	**Unit 2** **Sequence**
Study Skills	Application of transformation formulae, solve, explain and analyse	Analysis, critical thinking
Revision	Revision of provided exercises and end unit assessment	Revision of provided exercises and end unit assessment
Assessment	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.
Learning Outcomes	• Use transformation formula to simplify the trigonometric expressions. • Analyse and discuss the solution of trigonometric inequalities. • Solve problems involving trigonometry concepts. • The relationship between trigonometry and other subjects.	• Define a sequence and determine if a given sequence increases or decreases, converges or does not. • Determine the "nth" term and the sum of the first "n"terms of arithmetic progressions or geometric progression. • Apply the concepts of sequences to solve problems involving arithmetic or geometric sequences. • Relationship between the sequences and other subjects.

	Unit 3 Logarithmic and Exponential Functions	Unit 4 Solving Equations by Numerical Method
Number of Periods	14 + homework	21 + homework
Introduction	People such as scientists, sociologists and town planners are often more concerned with the rate at which a particular quantity is growing than with its current size. The director of education is more concerned with the rate of at which the school population is increasing or decreasing than with what the population is now, because he has to plan for the future and ensure that there are enough(and not too many) school places available to meet demand each year. The scientists may need to know the rate at which a colony of bacteria is growing rather than how many of the bacteria exists at this moment, or the rate at which a liquid is cooling rather than the temperature of the liquid now, or the rate at which a radioactive material is decaying rather than how many atoms current exist. The above events show us the areas where this unit finds use in our daily activities.	You should know how to solve linear equations and quadratic equations, either by factorizing or by completing the square. In some instances it may be almost impossible to use an exact method to solve equation, for example, $\theta - 1 - \sin\theta = 0$ precisely. In this unit we reconsider other techniques which give good approximations to the solution in a more formal way.
Classroom Organisation	Whole class orientation; then working in pairs or in groups.	Whole class orientation; then working in pairs or in groups.
Equipment Required	• Instruments of geometry • Scientific calculator	• Instruments of geometry • Scientific calculator

	Unit 3 **Logarithmic and Exponential Functions**	**Unit 4** **Solving Equations by Numerical Method**
Activities	Group work, practical	Group work, practical
Competences Practiced	Drawing, Creativity	Team work, data analysis, Critical thinking.
Language Practice	Presenting result obtained in activities.	Presenting result obtained in activities.
Vocabulary Acquisition	Exponential growth and decay	Interpolation, extrapolation
Study Skills	• Define logarithms and exponentials. • Properties of logarithms and exponentials and their applications. • To model and solve problem involving logarithms • Convert the logarithm to exponential form.	• Locate the roots of equation. • Approximate solutions of equations. • Selection of numerical method appropriate to a given problem.
Revision	Revision of provided exercises and end unit assessment	Revision of provided exercises and end unit assessment
Assessment	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.
Learning Outcomes	• State and demonstrate properties of logarithms and exponentials. • Carry out operations using the change of base of logarithms. • Apply logarithms or exponential to solve problem involving logarithms such radioactive-decay problems, Carbon dating problems, …	• Use numerical methods to approximate solutions of equations. • Select a numerical method appropriate to a given problem.

	Unit 5 Trigonometric and Inverse Trigonometric Functions	Unit 6 Vector Space of Real Numbers
Number of Periods	37 + homework	21+ homework
Introduction	Trigonometry has become an essential part of today's world. There are so many places where trigonometry is used; it is impossible to list them all. For example, there would be no space travel, or even air travel without trigonometry.	A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied by numbers, called scalars in this context. In physics, vectors are often used to describe forces. Classical Mechanics: Block sliding down a ramp: You need to calculate the force of gravity (a vector down), the normal force (a vector perpendicular to the ramp), and a friction force (a vector opposite the direction of motion).
Classroom Organisation	Whole class orientation; then working in groups, Research	Research, Whole class orientation; then working in groups.
Equipment Required	Geometric instruments and graph paper Scientific calculator Manila papers and markers	• Geometric instruments and graph paper. • Scientific calculator. • Manila papers and markers.
Activities	Group work, pairing, research	Group work, pairing, practical, research.
Competences Practiced	Numeracy, Drawing and creativity	Creativity
Language Practice	Presenting result obtained in activities.	Presenting result obtained in activities.

	Unit 5 **Trigonometric and Inverse Trigonometric Functions**	**Unit 6** **Vector Space of Real Numbers**
Vocabulary Acquisition	• Restricted domain • aperiodic	Geometric interpretation of dot product and cross product.
Study Skills	Solving and discussion	Calculation, discussion, explanation.
Revision	Revision of provided exercises and end unit assessment.	Revision of provided exercises and end unit assessment.
Assessment	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.
Learning Outcomes	• Extend the concepts of function, domain, range, period, inverse function, limits to trigonometric functions. • Solve problems involving trigonometric functions. • Calculation of limits of trigonometric functions and remove their indeterminate forms. • Calculation of high derivatives of trigonometric functions.	• A basis and the dimension of a vector space and give examples of bases of $\mathbb{R}^3$. • Operations of vectors of $\mathbb{R}^3$. • Dot product and the cross product of two vectors in a three dimensional vector space and their properties. • Magnitude of a three-dimensional vector. • Explain geometrically the dot product and the cross product.

	Unit 7 **Matrices and Determinant of Order 3**	**Unit 8** **Points, Straight Lines and Sphere in 3 D**
Number of Periods	35+ homework	34+ homework
Introduction	Matrices play a vital role in the projection of a three dimensional image into a two dimensional image. Matrices and their inverse are used by programmers for coding or encrypting a message. Matrices are applied in the study of electrical circuits, quantum mechanics and optics. A message is made as a sequence of numbers in a binary format for communication and it follows code theory for solving. Hence, with the help of matrices, those equations are solved. Matrices are used for taking seismic surveys.	In plane, the position of a point is determined by two numbers $x \; y$, obtained with reference to two straight lines in the plane intersecting at right angles. The position of point in space is, however, determined by three numbers x, y, z. The physical world in which we live is in three dimensional because through any point they can pass three, and no more, straight lines that are mutually perpendicular. That is, equivalent to the fact that we require three numbers to locate a point in space with respect to some reference point (origin).
Classroom Organisation	Whole class orientation; then working in groups	Home work, Whole class orientation; then working in groups.
Equipment Required	Scientific calculator	• Geometric instruments and graph paper. • Scientific calculator. • Manila papers and markers.
Activities	Group work, pairing	Group work, pairing, practical
Competences Practiced	Creativity, data analysis, communication skills, solve	Drawing, Analysis

	Unit 7 **Matrices and Determinant of Order 3**	**Unit 8** **Points, Straight Lines and Sphere in 3 D**
Language Practice	Presenting result obtained in activities.	Presenting result obtained in activities.
Vocabulary Acquisition	• Upper and lower triangular matrices. • Adjoint matrix.	Line and plane in space Sphere.
Study Skills	Solving, modeling and creativity.	Calculation, Analysis, explanation, discussion.
Revision	Revision of provided exercises and end unit assessment.	Revision of provided exercises and end unit assessment.
Assessment	• Formative assessments through activities • Summative assessments through exercises and end of unit assessments.	• Formative assessments through activities • Summative assessments through exercises and end of unit assessments.
Learning Outcomes	• Define operations on matrices of order 3. • Apply the properties on calculation of determinant of order 3. • Identify invertible matrix and determine inverse of matrix of order 3. • Determine a matrix from a linear transformation in 3D • Define and perform various operations on linear transformation of $\mathbb{R}^3$. • Reorganise data using matrices. • Apply matrices in solving related problems .	• Define the position of point and represent it in 3D. • Determine equations of a line in 3D. • Calculate the distance between two points in 3D. • Determine equations of plane in 3D. • Solve problems related to equations of line and plane. • Determine the diameter, radius and centre from equation of sphere. • Determine intersection of a sphere and plane; intersection of a sphere and a line.

	Unit 9 **Bivariate statistics**	**Unit 10** **Conditional Probability and Bayes Theorem**
Number of Periods	10+ homework	20+ homework
Introduction	**In descriptive statistics** data may be **qualitative** such as sex, color and so on or **quantitative** represented by numerical quantity such as height, mass, time and so on. The measures used to describe the data are measures of central tendency and measures of variability or dispersion. Until now, we know how to determine the measures of central tendency in one variable. In this chapter, we will use those measures in two quantitative variables known as **double series.**	**Probability** is a common sense for scholars and people in modern days. It is the chance that something will happen-how likely it is that some event will happen. No engineer or scientist can conduct research and development works without knowing the probability theory. Some academic fields based on the probability theory are statistics, communication theory, computer performance evaluation, signal and image processing, game theory.... Some applications of the probability theory are character recognition, speech recognition, opinion survey, missile control, seismic analysis...
Classroom Organisation	Homework, Whole class orientation; then working in groups.	Homework, Whole class orientation; then working in groups.
Equipment Required	• Instruments of geometry • Scientific calculator	Scientific calculator
Activities	Group work, pairing, research	Group work, pairing
Competences Practiced	Analysis, discussion	Analysis, discussion

	Unit 9 Bivariate statistics	Unit 10 Conditional Probability and Bayes Theorem
Language Practice	Presenting result obtained in activities.	Presenting result obtained in activities.
Vocabulary Acquisition	Bivariate data, covariance, coefficient of correlation, scatter diagram, regression line	Independent events
Study Skills	Explanation, analyse, calculation, critical thinking	Calculation, explanation, analysis
Revision	Revision of provided exercises and end unit assessment.	Revision of provided exercises and end unit assessment.
Assessment	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.	• Formative assessments through activities. • Summative assessments through exercises and end of unit assessments.
Learning Outcomes	• Plot visually data on scatter diagram and represent correlation between two variables. • Calculate covariance and determine coefficient of correlation. • Analyse and interpret data critically.	• Calculate number possible outcomes of occurring independent events under equally likely assumptions. • Compute the probability of an event B occurring when event A has already taken place. • To use correctly Bayes's theorem in solving problems.

Section 6: Example of Lesson plan

School: ……………………… Academic year: ……….

Teacher's name: ……………

Term	Date	Subject	Class	Unit No	Lesson No	Duration	Class size
1	……….	Mathematics	S5 MEG	1	……….	80 minutes	35

Type of Special Educational Needs and number of learners

For low vision learners: to avail big printed documents and facilitate these learners. Avoid making a group of low vision only otherwise it can be considered as segregation.

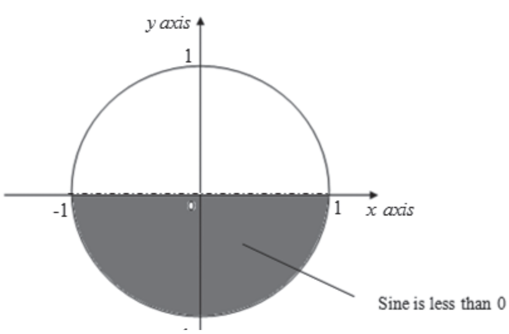
Gifted learners: to encourage them to explain, to each other and help their classmates.

Unit title	TRIGONOMETRIC FORMULAE AND EQUATIONS
Key Unit Competence:	Solve trigonometric equations, inequalities and related problems using trigonometric functions and equations.
Title of the lesson	Trigonometric inequalities
Instructional objective	Given instruments of geometry, learners should be able to represent a trigonometric inequality on trigonometric circle and find the solution accurately.
Plan for this Class	Location: Classroom Learners are organised into groups.
Learning Materials	Exercise book, pen, calculator, ruler
References	Learners' Book

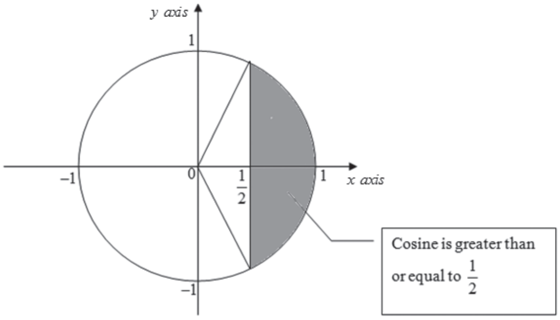
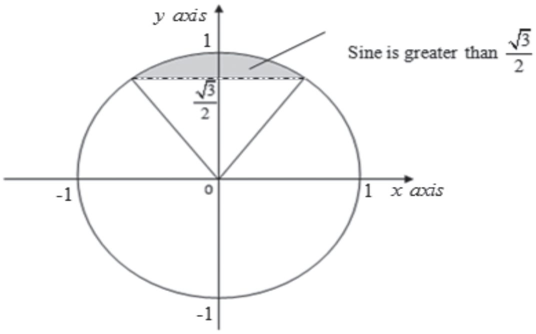
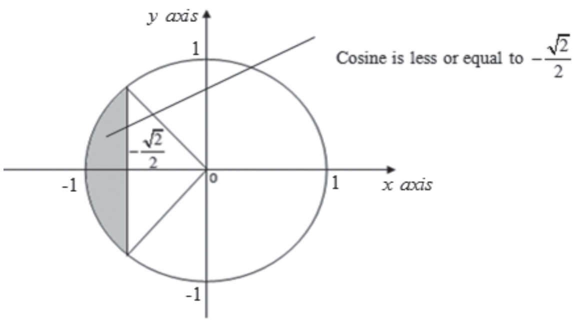
Description of teaching and learning activity

In groups, learners will do the activity 1.9 from learner's book page 23, make presentation of group findings. In conclusion, learners will do questions 1 and 2 of exercise 1.9 from learner's book page 30 in their respective groups and solve them on chalkboard. Learners will do question 3 of exercise 1.9 as individual quiz and question 4 will be an assignment. At the end of the lesson learners are also given another assignment to be discussed as an activity of the next lesson "**Application: Simple harmonic motion**".

Timing for each step	Teacher's activities	
Introduction **5 minutes**	Ask questions on previous lesson.	
Body of the lesson **15 minutes**	**Step 1:** Form groups • Request the learners to do activity 1.9 from learner's book page 23 in their groups • Goes round to check the progress of the discussion, and intervenes where necessary. • Guides learners with special educational needs on how to do activity.	
10 minutes	**Step 2:** Request a reporter from each group to present the work on the chalkboard.	

	Learners' activities	Competences and cross cutting issues to be addressed
	Question: Solve the following equations $\cos x = \frac{\sqrt{2}}{2}$ $\sin 2x = \frac{1}{2}$ Respond to questions on the chalkboard Answers: $x = \pm\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$ $x = \begin{cases} \frac{\pi}{12} + k\pi \\ \frac{5\pi}{12} + k\pi \end{cases},\ k \in \mathbb{Z}$	Students are developing communication skills when they are explaining and sharing ideas
	In their groups, learners will do activity 1.9 • Draw a trigonometric circle • Represent the given inequality on trigonometric circle. • Reporter represents the work. • Learners interact through questions and comments. **Answers:** sine is less than 0 	• Cooperation and interpersonal management developed through working in groups . • Communication: learners communicate and convey information and ideas through speaking when they are presenting their work. • Self confidence: learners will gain self confidence competence when they are presenting their work.

10 minutes	**Step 3:** Capture the main points from the presentation of the learners and summarise them.	

	cosine is greater or equal to $\frac{1}{2}$  sine is greater than $\frac{\sqrt{3}}{2}$  cosine is less or equal to $-\frac{\sqrt{2}}{2}$ 	• In group activities, the fact of being convinced without fighting, peace and education values are developed too.

Conclusion **10 minutes**	Request learners to give the main points of the learned lesson in summary.	
15 minutes	Request learners to do exercise 1.9 in their respective groups. Goes round to check the progress of the discussion, and intervenes where necessary.	
15 minutes	Request some learners to answer to questions 1 and 2 of exercise 1.9 on chalkboard. Ensures that the learners understood the learned lesson and decide to repeat the lesson or to continue with new lesson next time	
10 minutes	Give to the learners an individual evaluation (quiz) and homework to the leaned lesson. **Lead into next lesson** Request learners to do activity 1.10 at home.	
Teacher self evaluation	Even if the objective has been achieved, some learners had no instruments of geometry and calculators. The time management has been disturbed by the fact of borrowing materials from their classmates. For this reason, next time each learner must have his/her own materials (instruments of geometry, calculator, …)	

	Summarize the learned lesson When solving inequalities, first replace the inequality sign by equal sign and then solve. Find all no equivalent angles in $[0,2\pi]$. Place these angles on a trigonometric circle. They will divide the circle into arcs. Choose the arcs containing the angles corresponding to the given inequality. Do questions 1 and 2 of exercise 1.9, from learner's book page 30, in their respective groups. Do questions 1 and 2 of exercise 1.9, from learner's book page 30, on chalkboard. Do question 3 of exercise 1.9, from learner's book page 30, as individual quiz. Do the given quiz individually	Learners develop critical thinking through generating a summary. Through group activities, cooperation is developed. Through presentation on chalkboard, communication skills are developed

Section 7: General Methodology

Follow the following three steps when teaching any lesson.

Introduction

Reviews previous lesson through asking the learners some questions. If there is no previous lesson, ask them pre-knowledge questions on the day lesson.

Body of the lesson

Give an activity to learners that will be done in groups or individually. Invite one or more groups for presentation of their work to other groups. If the activity is individual, ask one or more learners to present his/her work to others. After activities, capture the main points from the presentation of the learners and summarise them.

Conclusion

Ask learners what they learned in day lesson. Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give homework to the learners. Give them two home works: one for the lesson of the day and another which will be activity for the next lesson.

Section 8: Description of Units

Unit 1. Trigonometric formulae, Equations and Inequalities

Unit 2. Sequences

Unit 3. Logarithmic and Exponential Equations

Unit 4. Solving Equations by Numerical Methods

Unit 5. Trigonometric Functions and their Inverses

Unit 6. Matrices and Determinant of Order 3

Unit 7. Vector Space of Real Numbers

Unit 8. Points, Straight lines, Planes and Sphere in 3D

Unit 9. Bivariate Statistics

Unit 10. Conditional Probability and Bayes theorem

Unit 1 Trigonometric Formulae, Equations and Inequalities

Learner's Book page 1 – 38

Aim

Solve trigonometric equations and related problems using trigonometric functions and equations

Objectives

After completing this unit, the learners should be able to:

- Use trigonometric formulae
- Solve trigonometric equations
- Solve trigonometric inequalities

Materials

Exercise books, pens, instruments of geometry, calculator

Contents

1.1. Trigonometric formulae

Recommended teaching periods: 14 periods

This section looks at **trigonometric formulae**

- Addition and subtraction formulae (compound formulae)

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot y+\cot x}$$

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

- Double angles

$\cos^2 x=1-\sin^2 x$ and $\sin^2 x=1-\cos^2 x$

$$\sin 2x=2\sin x\cos x$$

$$\cos 2x=\cos^2 x-\sin^2 x$$

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

$$\cot 2x=\frac{\cot^2 x-1}{2\cot x}$$

- Half angle formulae

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2}=\pm\sqrt{\frac{\cos x+1}{2}}$$

$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$ or $\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}$

- Transformation of product in sum

$$\cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x\sin y=-\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$$

$$\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

- Transformation of sum in product formulae

$$\cos p + \cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

- t-Formulae

If $t = \tan\frac{A}{2}$, then $\sin A = \frac{2t}{1+t^2}, \cos A = \frac{1-t^2}{1+t^2}, \tan A = \frac{1+t^2}{1-t^2}$

Teaching guidelines

Let learners know basic relations in trigonometry like

$$\sec x = \frac{1}{x},\ \csc x = \frac{1}{\sin x},\ \tan x = \frac{\sin x}{\cos x},\ \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}.$$

Help them to recall those basic relations.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.

- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 1.1 Page 2

Materials

Exercise book, pens

Answers

a) $$\sin(A+B)=\frac{PR}{OP}$$
$$=\frac{PQ+QR}{OP}$$
$$=\frac{PQ+TS}{OP}$$
$$=\frac{PQ}{OP}+\frac{TS}{OP}$$
$$=\left(\frac{PQ}{PT}\times\frac{PT}{OP}\right)+\left(\frac{TS}{OT}\times\frac{OT}{OP}\right)$$
$$=\cos A\sin B+\sin A\cos B$$

So, $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

b) Similarly, $$\cos(A+B)=\frac{OR}{OP}$$
$$=\frac{OS-RS}{OP}$$
$$=\frac{OS-QT}{OP}$$
$$=\frac{OS}{OP}-\frac{QT}{OP}$$
$$=\left(\frac{OS}{OT}\times\frac{OT}{OP}\right)-\left(\frac{QT}{PT}\times\frac{PT}{OP}\right)$$
$$=\cos A\cos B-\cos A\cos B$$

Thus, $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

Now, $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

From the identities for $\sin(A+B)$ and $\cos(A+B)$, you have $\tan(A+B)=\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$$

$$=\frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}}+\dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\ \cancel{\cos B}}{\cancel{\cos A}\ \cancel{\cos B}}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

So, $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

- Replacing B by $-B$ in the identity for $\sin(A+B)$ gives

  $\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B)$

  Or $\sin(A-B)=\sin A\cos B-\cos A\sin B$

- Replacing B by $-B$ in the identity for $\cos(A+B)$ gives

  $\cos(A-B)=\cos A\cos(-B)-\sin A\sin(-B)$.

  Thus,

  $\cos(A-B)=\cos A\cos B+\sin A\sin B$

- Replacing B by in the identity for $\tan(A+B)$ yields

  $\tan(A-B)=\dfrac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$

  Hence, $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

Exercise 1.1 Page 4

1. $$2\sin\theta\sin 4\theta + 2\cos\theta\cos 4\theta = 2(\sin\theta\sin 4\theta + \cos\theta\cos 4\theta)$$
$$= 2\cos(4\theta - \theta) = 2\cos 3\theta$$

2. a) $$\sin 75^0 = \sin(45^0 + 30^0)$$
$$= \sin 45^0 \cos 30^0 + \cos 45^0 \sin 30^0$$
$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

b) $$\cos\frac{13\pi}{6} = \cos\left(2\pi + \frac{\pi}{6}\right)$$
$$= \cos 2\pi \cos\frac{\pi}{6} - \sin 2\pi \sin\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$

c) $$\tan 330^0 = \tan(360^0 - 30^0)$$
$$= \frac{\tan 360^0 - \tan 30^0}{1 + \tan 360^0 \tan 30^0} = \frac{0 - \frac{\sqrt{3}}{3}}{1} = -\frac{\sqrt{3}}{3}$$

5. a) $2+\sqrt{3}$ b) $\frac{\sqrt{6}+\sqrt{2}}{4}$ c) $\frac{\sqrt{3}}{2}$

d) $\frac{1}{2}$ e) -1

Activity 1.2 Page 5

Materials

Exercise book, pens

Answers

1. $$\cos(x+x) = \cos x\cos x - \sin x\sin x$$
$$\Rightarrow \cos 2x = \cos^2 x - \sin^2 x$$

2. $$\cos(x-x) = \cos x\cos x + \sin x\sin x$$
$$\Leftrightarrow \cos 0 = \cos^2 x + \sin^2 x$$

$$\Leftrightarrow 1 = \cos^2 x + \sin^2 x$$

$$\Rightarrow \cos^2 x + \sin^2 x = 1$$

3. $\sin(x + x) = \sin x \cos x + \cos x \sin x$

$$\Rightarrow \sin 2x = 2\sin x \cos x$$

4. $\tan(x + x) = \dfrac{\tan x + \tan x}{1 - \tan x \tan x}$

$$\Rightarrow \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

5. $\cot(x + x) = \dfrac{\cot x \cot x - 1}{\cot x + \cot x}$

$$\Rightarrow \cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

Exercise 1.2 Page 7

1. $4\sin x \cos^3 x - 4\cos x \sin^3 x$
2. $\cos^8 x + \sin^8 x - 28\cos^2 x \sin^6 x + 70\cos^4 x \sin^4 x - 28\cos^6 x \sin^2 x$
3. $2\sin 15^0 \cos 15^0 = \sin\left(2 \times 15^0\right) = \sin 30^0 = \dfrac{1}{2}$
4. $\dfrac{1}{\sqrt{2}}$
5. a) $\dfrac{4}{5}, \dfrac{3}{5}, \dfrac{4}{3}$ b) $-\dfrac{4}{5}, \dfrac{3}{5}, -\dfrac{4}{3}$

Activity 1.3 Page 7

Materials

Exercise book, pens

Answer

From the double angle formulae, you have

1. $\cos 2x = \cos^2 x - \sin^2 x$

$= \left(1 - \sin^2 x\right) - \sin^2 x$ from $\cos^2 x + \sin^2 x = 1$

$= 1 - 2\sin^2 x$

So, $\cos 2x = 1 - 2\sin\ x$

Letting $\theta = 2x$, $\cos 2x = 1 - 2\sin^2 x$ gives

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

Or $2\sin^2\frac{\theta}{2} = 1 - \cos\theta \Rightarrow \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

So, $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

2. $\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2 x - \left(1 - \cos^2 x\right)$ from $\cos^2 x + \sin^2 x = 1$

$= 2\cos^2 x - 1$

So, $\cos 2x = 2\cos^2 x - 1$

Letting $\theta = 2x$, $\cos 2x = 2\cos^2 x - 1$ gives

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 \Leftrightarrow 2\cos^2\frac{\theta}{2} = 1 + \cos\theta$$

$$\cos^2\frac{\theta}{2} = \frac{1}{2}\left(1 + \cos\theta\right) \Rightarrow \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Thus, $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

3. $\tan\frac{\theta}{2} = \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$$\Leftrightarrow \tan\frac{\theta}{2} = \frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}} \qquad \Leftrightarrow \tan\frac{\theta}{2} = \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$$

By rationalizing denominator, you get

$$\tan\frac{\theta}{2} = \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\,\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{1-\cos^2\theta}} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{|1-\cos\theta|}{\sqrt{1-\cos^2\theta}}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{\sqrt{1-\cos^2\theta}} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{|\sin\theta|}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

So, $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$

From $\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$, conjugating numerator, you get

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{1-\cos^2\theta}}{\sqrt{(1+\cos\theta)^2}} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{\sin^2\theta}}{\sqrt{(1+\cos\theta)^2}}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{|\sin\theta|}{|1-\cos\theta|} \qquad \Leftrightarrow \tan\frac{\theta}{2}=\frac{|\sin\theta|}{|1+\cos\theta|}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

So $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$

Therefore, $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$ or $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$

Exercise 1.3 Page 8

1. If $\cos A = -\frac{7}{25}$,

$$\sin\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}} = \pm\sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}} = \pm\sqrt{\frac{\frac{32}{25}}{2}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5};$$

$$\cos\frac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}} = \pm\sqrt{\frac{1-\frac{7}{25}}{2}} = \pm\sqrt{\frac{\frac{18}{25}}{2}} = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5};$$

$$\tan A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \pm\sqrt{\frac{1+\frac{7}{25}}{1-\frac{7}{25}}} = \pm\sqrt{\frac{32}{18}} = \pm\frac{4}{3}$$

2. If $\tan 2A = \frac{7}{24}$, $0 < A < \frac{\pi}{4}$, to find $\tan A$

$$\tan 2A = \frac{2\tan A}{1-tan^2 A}$$

$$\Rightarrow \frac{7}{24} = \frac{2\tan A}{1-tan^2 A}$$

$$\Rightarrow 7 - 7tan^2 A = 48\tan A$$

$$\Rightarrow 7tan^2 A - 48\tan A - 7 = 0$$

$$\Rightarrow (7\tan A - 1)(\tan A + 7) = 0$$

$\Rightarrow \tan A = \frac{1}{7}$ since $\tan A = 7$ is impossible for $0 < A < \frac{\pi}{4}$

3. $\frac{\sqrt{2-\sqrt{2}}}{2}, \frac{\sqrt{2+\sqrt{2}}}{2}, 1-\frac{\sqrt{2}}{2}$

4. $\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$

Activity 1.4 Page 9

Materials

Exercise book, pens

Answer

1. $\sin(x+y)+\sin(x-y)=\sin x\cos y+\cancel{\cos x\sin y}+\sin x\cos y-\cancel{\cos x\sin y}$
$=2\sin x\cos y$

2. $\sin(x+y)-\sin(x-y)=\sin x\cos y+\cos x\sin y-(\sin x\cos y-\cos x\sin y)$
$=\cancel{\sin x\cos y}+\cos x\sin y-\cancel{\sin x\cos y}+\cos x\sin y$
$=2\cos x\sin y$

3. $\cos(x+y)+\cos(x-y)=\cos x\cos y-\cancel{\sin x\sin y}+\cos x\cos y+\cancel{\sin x\sin y}$
$=2\cos x\cos y$

4. $\cos(x+y)-\cos(x-y)=\cos x\cos y-\sin x\sin y-(\cos x\cos y+\sin x\sin y)$
$=\cancel{\cos x\cos y}-\sin x\sin y-\cancel{\cos x\cos y}-\sin x\sin y$
$=-2\sin x\sin y$

Exercise 1.4 Page 10

1. a) $\sin x\cos 3x=\frac{1}{2}(\sin 4x-\sin 2x)$

b) $\cos 12x\sin 9x=\frac{1}{2}(\sin 21x-\sin 3x)$

c) $-\frac{1}{2}(\cos 20x-\cos 2x)$

d) $\sin 8x-\sin 2x$

e) $\frac{1}{2}(\cos 4x+\cos x)$

Activity 1.5 Page 10

Materials

Exercise book, pens

Answers

The formulae for transforming product in sum are

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right] \quad \text{(Equation 1)}$$

$$\sin x \sin y = -\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right] \quad \text{(Equation 2)}$$

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right] \quad \text{(Equation 3)}$$

$$\cos x \sin y = \frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right] \quad \text{(Equation 4)}$$

$$\begin{cases} x+y=p \\ x-y=q \end{cases} \Rightarrow \begin{cases} x=\dfrac{p+q}{2} \\ y=\dfrac{p-q}{2} \end{cases} \quad \text{(i)}$$

From (i)

Equation (1) becomes

$$\cos\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\cos p+\cos q)$$

Equation (2) becomes

$$\sin\frac{p+q}{2}\sin\frac{p-q}{2}=-\frac{1}{2}(\cos p-\cos q)$$

Equation (3) becomes

$$\sin\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\sin p+\sin q)$$

Equation (4) becomes

$$\cos\frac{p+q}{2}\sin\frac{p-q}{2}=\frac{1}{2}(\sin p-\sin q)$$

Exercise 1.5 Page 11

1. a) $\cos x+\cos 7x = 2\cos 4x\cos 3x$

   b) $\sin 4x-\sin 9x = -2\cos\frac{13x}{2}\sin\frac{5x}{2}$

   c) $\sin 3x+\sin x = 2\sin 2x\cos x$

   d) $\cos 2x-\cos 4x = 2\sin 3x\sin x$

1.2. Trigonometric equations

Recommended teaching periods: 7 periods

The solutions of a trigonometric equation for which $0 \le x \le 2\pi$ are called **principle solutions** while the expression (involving integer k) of solution containing all values of the unknown angle is called the **general solution** of the trigonometric equation. When the interval of solution is not given, you are required to find general solution.

When solving trigonometric equation, **note that general solution for**

- $\sin x = 0$ is $x = k\pi, k \in \mathbb{Z}$
- $\cos x = 0$ is $x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$
- $\tan x = 0$ is $x = k\pi, k \in \mathbb{Z}$
- all angles having the same sine i.e. $\sin x = \sin\theta$ is $x = (-1)^k \theta + k\pi, k \in \mathbb{Z}$
- all angles having the same cosine i.e. $\cos x = \cos\theta$ is $x = \pm\theta + 2k\pi, k \in \mathbb{Z}$
- all angles having the same tangent i.e. $\tan x = \tan\theta$ is $x = \theta + k\pi, k \in \mathbb{Z}$

The sum or difference of trigonometric functions containing unknown are transformed into the sum.

Remember that

$$\cos p + \cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

To find the general solution, the equation of the form

$a\sin x + b\cos x = c$ where $a, b, \mathrm{c} \in \mathbb{Z}$ such that $|c| \le \sqrt{a^2+b^2}$

a) Divide each term by $\sqrt{a^2+b^2}$ and convert it in the form

$$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}$$

b) Let $\tan\theta = \frac{b}{a}$, then $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}, \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$.

The given equation reduces to the form

$r\cos\theta\cos x + r\sin\theta\sin x = c$ or $\cos\theta\cos x + \sin\theta\sin x = \frac{c}{r}$

c) Then, $\cos(x-\theta) = \cos\alpha$,where $\cos\alpha = \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}}$

d) Therefore, $x = \pm\alpha + \theta + 2k\pi, k \in \mathbb{Z}$

Alternative method: in $a\sin x + b\cos x = c$

Using t-formula, $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$, where $t = \tan\frac{x}{2}$, gives

$$a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2} = c \Rightarrow 2at + b - bt^2 = c\left(1+t^2\right)$$

$\Leftrightarrow (b+c)t^2 - 2at + c - b = 0$ which is quadratic equation in t.

Remember that $t = \tan\frac{x}{2}$.

Notice

In solving the trigonometric equation, it is helpful to remember the following identities:

$\sin\alpha = \sin(\alpha + 2k\pi),\ k \in \mathbb{Z}$	$\sin\alpha = \sin(\pi - \alpha)$
$\cos\alpha = \cos(\alpha + 2k\pi),\ k \in \mathbb{Z}$	$\cos\alpha = \cos(-\alpha)$
$\tan\alpha = \tan(\alpha + k\pi),\ k \in \mathbb{Z}$	$\tan\alpha = \tan(\alpha + \pi)$

Teaching guidelines

Let learners know what inverse function is. Help them to recall that $f^{-1}\left[f\left(x\right)\right]=x$. Make sure that learners have scientific calculators.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 1.6 Page 11

Materials

Exercise book, pens and calculator

Answers

1. $\begin{cases}\dfrac{\pi}{6}+2k\pi \\ \dfrac{5\pi}{6}+2k\pi\end{cases},\ k\in\mathbb{Z}$

2. $\pm\dfrac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$

3. $\dfrac{\pi}{6}+k\pi,\ k\in\mathbb{Z}$

Exercise 1.6 Page 16

1. a) $\frac{\pi}{3}, \frac{2\pi}{3}$ b) $\frac{2\pi}{3}, \frac{5\pi}{3}$ c) $\frac{\pi}{3}, \frac{5\pi}{3}$

   d) $\frac{\pi}{6}$ e) $\frac{\pi}{3}, \frac{2\pi}{3}$

2. a) $\frac{5\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$ b) $-\frac{\pi}{3} + k\pi,\ k \in \mathbb{Z}$

   c) $\frac{\pi}{3} + k\pi, \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$ d) $k\pi, \pm\frac{\pi}{6} + k\pi,\ k \in \mathbb{Z}$

Activity 1.7 Page 17

Materials

Exercise book, pens and calculator

Answers

a) $\cos 2x = \frac{1}{\sqrt{2}}$ is positive, thus, $2x$ lies in the 1st or 4th quadrant.

$$\cos 2x = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \quad \text{or} \quad \cos\left(2\pi - \frac{\pi}{4}\right)$$

$$\Rightarrow 2x = \frac{\pi}{4} + 2k\pi \ \ or\ \ 2x = \frac{7\pi}{4} + 2k\pi, k \in \mathbb{Z}$$

General solution of $\cos 2x = \frac{1}{\sqrt{2}}$ is $\frac{\pi}{8} + k\pi,\ k \in \mathbb{Z}$ or

$$x = \frac{7\pi}{8} + k\pi,\ k \in \mathbb{Z}$$

b) $\sin\frac{x}{2} = -\frac{1}{2}$ is negative $\Rightarrow \frac{x}{2}$ lies in the 3rd or 4th quadrant.

Here, $\sin\frac{x}{2} = -\frac{1}{2} = \sin\left(\pi + \frac{\pi}{6}\right) \quad \text{or} \quad \sin\left(-\frac{\pi}{6}\right)$

$$\Rightarrow \frac{x}{2} = \frac{7\pi}{6} \ \ or\ \ \frac{x}{2} = -\frac{\pi}{6}$$

The general solution of $\sin\frac{x}{2}=-\frac{1}{2}$ is $\frac{x}{2}=-\frac{\pi}{6}+2k\pi$ or $\frac{7\pi}{6}+2k\pi, k\in\mathbb{Z}$

$\Rightarrow x=-\frac{\pi}{3}+4k\pi$ or $\frac{7\pi}{3}+4k\pi\equiv\frac{\pi}{3}+4\mathrm{k}\,\pi, k\in\mathbb{Z}$

c) $\sin mx+\sin nx=0\Leftrightarrow 2\sin\frac{(m+n)x}{2}\cos\frac{(m-n)x}{2}=0$

$$\Rightarrow \sin\frac{(m+n)x}{2}=0\ or\ \cos\frac{(m-n)x}{2}=0$$

$$\Rightarrow \frac{(m+n)x}{2}=k\pi\ or\ \frac{(m-n)x}{2}=\frac{\pi}{2}+k\pi$$

$$\Rightarrow x=\frac{2k\pi}{m+n}\ or\ \frac{(m-n)x}{2}=\frac{\pi}{2}(2k\pi+1)$$

$$\Rightarrow x=\frac{2k\pi}{m+n}\ or\ x=\frac{(2k\pi+1)\pi}{m-n}$$

General solution is $x=\frac{2k\pi}{m+n}\ or\ x=\frac{(2k\pi+1)\pi}{m-n}$

d) $\cos 4x-\cos 2x=0\Leftrightarrow 2\sin 3x\sin x=0$

$$\Rightarrow \sin 3x=0\ or\ \sin x=0$$

$$\Rightarrow 3x=k\pi\ or\ x=k\pi$$

$$\Rightarrow x=\frac{k\pi}{3}\ or\ x=k\pi$$

General solution is $x=\frac{k\pi}{3}\ or\ x=k\pi$

Exercise 1.7 Page 21

1.	$\pm\frac{\pi}{12}+\frac{k\pi}{4},\ k\in\mathbb{Z}$	2.	$\left\{0,\frac{\pi}{14},\frac{\pi}{3}\right\}$
3.	$\{30^0,150^0,199.5^0,340.5^0\}$	4.	$\{170.7^0,350.7^0\}$
5.	$\frac{k\pi}{2}\ or\ \frac{\pi}{2}(2k\pi+1),\ k\in\mathbb{Z}$	6.	$(2k+1)\frac{\pi}{2},\ (2k+1)\frac{\pi}{8}$

7. $\frac{k\pi}{3}, \pm\frac{\pi}{12}+\frac{k\pi}{2}, k\in\mathbb{Z}$

8. $(2k+1)\frac{\pi}{8}, (2k+1)\frac{\pi}{4}, (2k+1)\frac{\pi}{2}, k\in\mathbb{Z}$

Activity 1.8 Page 21

Materials

Exercise book, pens and calculator

Answers

$$\sqrt{3}\cos x-\sin x=\sqrt{3}$$

1. $\Rightarrow \cos x-\frac{\sin x}{\sqrt{3}}=1$

2. $\tan\alpha=-\frac{1}{\sqrt{3}}\Rightarrow\frac{\sin\alpha}{\cos\alpha}=-\frac{1}{\sqrt{3}}\ or\ \alpha=-\frac{\pi}{6}$

3. $\cos x-\frac{\sin x}{\sqrt{3}}=1\Rightarrow\cos x-\frac{\sin\alpha}{\cos\alpha}\sin x=1$

$$\Leftrightarrow\cos\alpha\cos x-\sin\alpha\sin x=\cos\alpha$$

$$\Leftrightarrow\cos(x-\alpha)=\cos\alpha$$

$$\Leftrightarrow\cos\left(x-\left(-\frac{\pi}{6}\right)\right)=\cos\left(-\frac{\pi}{6}\right)\quad\Leftrightarrow\cos\left(x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$\Rightarrow x+\frac{\pi}{6}=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow\begin{cases}x+\frac{\pi}{6}=-\frac{\pi}{6}+2k\pi\\ x+\frac{\pi}{6}=\frac{\pi}{6}+2k\pi\end{cases},\ k\in\mathbb{Z}\ \Rightarrow\begin{cases}x=-\frac{\pi}{3}+2k\pi\\ x=2k\pi\end{cases},\ k\in\mathbb{Z}$$

4. $\cos\alpha\cos x-\sin\alpha\sin x=\cos\alpha\Leftrightarrow\cos(x-\alpha)=\cos\alpha$

$$\Leftrightarrow\cos\left(x-\left(-\frac{\pi}{6}\right)\right)=\cos\left(-\frac{\pi}{6}\right)\qquad\Leftrightarrow\cos\left(x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$\Rightarrow x+\frac{\pi}{6}=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \begin{cases} x+\frac{\pi}{6}=-\frac{\pi}{6}+2k\pi \\ x+\frac{\pi}{6}=\frac{\pi}{6}+2k\pi \end{cases},\ k\in\mathbb{Z} \qquad \Rightarrow \begin{cases} x=-\frac{\pi}{3}+2k\pi \\ x=2k\pi \end{cases},\ k\in\mathbb{Z}$$

Exercise 1.8 Page 22

1. $\left\{x=\frac{\pi}{6}+k\pi, x=\frac{\pi}{2}, k\in\mathbb{Z}\right\}$
2. $\left\{x=\frac{\pi}{4}+k\pi, k\in\mathbb{Z}\right\}$
3. $\pm\frac{3\pi}{4}-\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$
4. $\frac{\pi}{6}\pm\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$
5. $\frac{\pi}{6}\pm\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}$
6. $-\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$

1.3. Trigonometric inequalities

Recommended teaching periods: 14 periods

To solve inequalities:

- First replace the inequality sign by equal sign and then solve.
- Find all no equivalent angles in $[0,2\pi]$.
- Place these angles on a trigonometric circle. They will divide the circle into arcs.
- Choose the arcs containing the angles corresponding to the given inequality.

Teaching guidelines

Let learners know what inverse function is and what trigonometric circle is. Help them to recall how to draw a trigonometric circle. Make sure that learners have mathematical instruments and scientific calculators.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

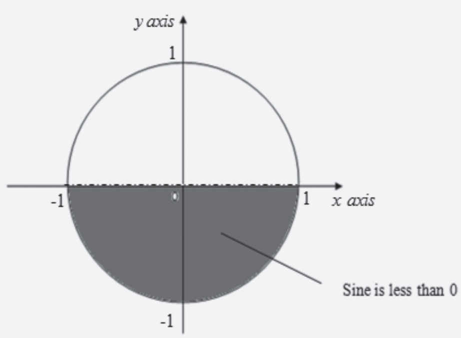
Activity 1.9 Page 23

Materials

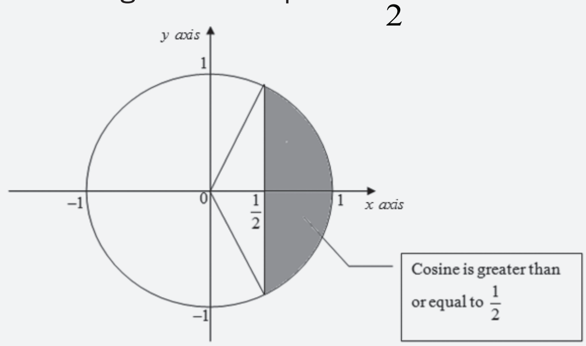
Exercise book, pens, instruments of geometry and calculator

Answers

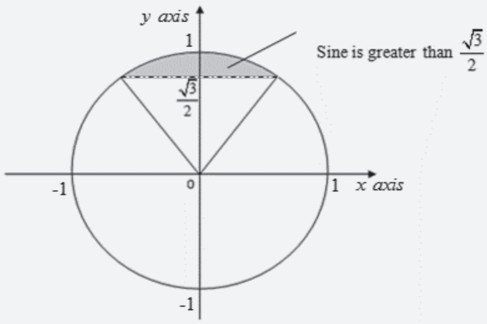
1. sine is less than 0



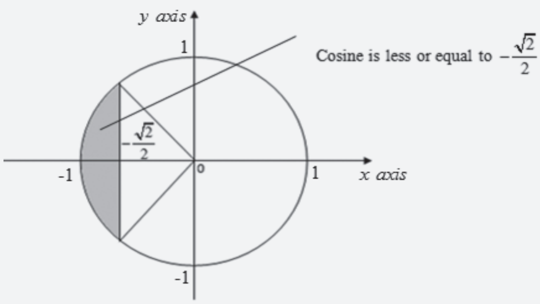
2. cosine is greater or equal to $\frac{1}{2}$



3. sine is greater than $\frac{\sqrt{3}}{2}$



4. cosine is less or equal to $-\frac{\sqrt{2}}{2}$



Exercise 1.9 Page 30

1. $\left[2k\pi, \frac{\pi}{3}+2k\pi\right] \cup \left[\frac{2\pi}{3}+2k\pi, (k+1)2\pi\right], \; k \in \mathbb{Z}$
2. $\left]2k\pi, \frac{\pi}{4}+2k\pi\right[\cup \left]\frac{3\pi}{4}+2k\pi, \pi+2k\pi\right[\cup \left]\frac{5\pi}{4}+2k\pi, \frac{7\pi}{4}+2k\pi\right[, \; k \in \mathbb{Z}$
3. $\left]\frac{7\pi}{18}+2k\pi, \frac{11\pi}{18}+2k\pi\right[\cup \left]\frac{19\pi}{18}+2k\pi, \frac{23\pi}{18}+2k\pi\right[\cup \left]\frac{31\pi}{18}\right.$
   $\left.+2k\pi, \frac{35\pi}{18}+2k\pi\right[, \; k \in \mathbb{Z}$
4. $k\pi, \; k \in \mathbb{Z}$

1.4. Applications

Activity 1.10 Page 30

Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on harmonic motion and how trigonometry is used in harmonic motion.

An object that moves on a coordinate axis is in simple harmonic motion if its distance from the origin, d, at time t is given by either $d = a\cos\omega t$ or $d = a\sin\omega t$. The motion has amplitude $|a|$, the maximum displacement of the object from its rest position. The period of the motion is $\frac{2\pi}{\omega}$, where $\omega > 0$.

Activity 1.11 Page 31

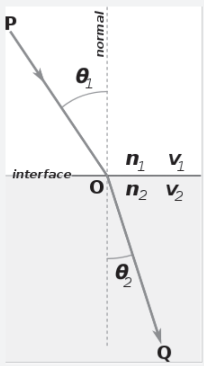
Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on Snell's law.

The degree of bending of the light's path depends on the angle that the incident beam of light makes with the surface, and on the ratio between the refractive indices of the two media.



Snell's law state that: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

End of Unit Assessment Page 36

1. $\dfrac{4\tan x-4\tan^3 x}{1-6\tan^2 x+\tan^4 x}$

2. $\dfrac{\cot^4 x-6\cot^2 x+1}{4\cot^3 x-4\tan x}$

3. $\sin 2\theta=\dfrac{120}{169}$

4. $2\cot 2a=\dfrac{2}{\tan 2a}$

$$=2\left(\frac{1-\tan^2 a}{2\tan a}\right)=2\left(\frac{1}{2\tan a}-\frac{\tan^2 a}{2\tan a}\right)$$

$$=\frac{1}{\tan a}-\tan a=\cot a-\tan a$$

5. a) $\dfrac{1}{2}$ b) $\dfrac{\sqrt{2}}{2}$

   c) -1 d) $\dfrac{\sqrt{3}}{2}$

6. $-\dfrac{7}{25}$

7. $-\dfrac{120}{119}$

8. a) $2\cos\dfrac{17}{2}\cos\dfrac{x}{2}$ b) $2\sin 7x\cos 4x$

9. a) $\dfrac{1}{2}(\sin 15x-\sin 7x)$ b) $\dfrac{1}{2}(\sin 16x+\sin 2x)$

10. a) $\dfrac{\pi}{6},\dfrac{\pi}{4},\dfrac{5\pi}{6}$ b) $\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{5\pi}{6}$

11) $169.6^0, 349.6^0$

12) $60^0, 240^0$

13) $126.2^0, 306.2^0$

14) $85.9^0, 265.9^0$

15) $22.5^0, 112.5^0, 202.5^0, 292.5^0$

16) $0^0, 60^0, 180^0, 300^0, 360^0$

17) $0^0, 120^0, 180^0, 240^0, 360^0$

18) $21.5^0, 158.5^0$

19) $0^0, 60^0, 120^0, 180^0, 240^0, 300^0$

20) $19.5^0, 160.5^0, 270^0$

21) $33.7^0, 63.4^0, 213.7^0, 243.4^0$

22) $33.7^0, 153.4^0, 213.7^0, 333.4^0$

23) $30^0, 90^0, 150^0, 270^0$

24) $120^0, 240^0$

25) $15^0, 75^0, 195^0, 255^0$

26) $0^0, 180^0, 360^0, 60^0, 300^0, 120^0, 240^0$

27) $0^0, 180^0, 360^0, 45^0, 135^0, 225^0, 315^0$

28) $0^0, 14.5^0, 165.5^0, \pm 180^0$

29) $\pm 180^0$

30) $0^0, \pm 48.2^0$

31) $19.5^0, 160.5^0$

32) $\pm 45^0, \pm 135^0$

33) $-63.4^0, 0, 116.6^0, \pm 180^0$

34) $60^0, 300^0$

35) $55.9^0, 145.9^0, 235.9^0, 325.9^0$

36) $60^0, 120^0, 240^0, 300^0$

37) $120^0, 240^0$

38) $26.6^0, 135^0, 206.6^0, 315^0$

39) $68.2^0, 135^0, 248.2^0, 315^0$

40) $194.5^0, 345.5^0$

41) $26.6^0, 45^0, 206.6^0, 255^0$

42) $70.5^0, 289.5^0$

43) $\left]\frac{\pi}{5}, \frac{2\pi}{5}\right[\cup \left]\frac{3\pi}{5}, \frac{4\pi}{5}\right[\cup \left]\pi, \frac{6\pi}{5}\right[\cup \left]\frac{7\pi}{5}, \frac{8\pi}{5}\right[\cup \left]\frac{7\pi}{5}, 2\pi\right[$

44) $\left[\frac{\pi}{18}, \frac{11\pi}{18}\right] \cup \left[\frac{13\pi}{18}, \frac{23\pi}{18}\right] \cup \left[\frac{25\pi}{18}, \frac{35\pi}{18}\right]$

45) $\left]\frac{\pi}{3}, \frac{2\pi}{3}\right[\cup \left]\frac{4\pi}{3}, \frac{5\pi}{3}\right[$

46) $10cm,\ 5cm,\ -5\sqrt{2}cm$

47) $3m,\ 4m,\ -3m$

Unit 2 Sequences

Learner's Book page 39 – 82

Aim

Understand, manipulate and use arithmetic, geometric sequences.

Objectives

After completing this unit, the learners should be able to:

- Define a sequence
- Determine whether a sequence converges or diverges
- Indicate the monotone sequences
- Determine harmonic sequences
- Identify arithmetic and geometric progressions and their properties
- Use sequences in daily life

Materials

Exercise books, pens, instruments of geometry, calculator

Contents

2.1. Generalities on sequences

Recommended teaching periods: 7 periods

This section looks at the definition of a **numerical sequence**, **convergence** and **divergence** sequences, **monotonic sequences**.

A sequence is a function whose domain is either $\mathbb{N}$ or subset of the form $\{1,2,3,4,\cdots,n\}$; depending on the domain of definition, a sequence is finite or infinite.

A numerical sequence $\{u_n\}$ is said to be convergent if it has a finite limit as $n \to \infty$ otherwise it is said to be divergent.

If $\lim_{x \to +\infty} u_n = L$ number L is called a limit of a numerical sequence

A sequence $\{u_n\}$ is said to be:

- **Increasing** or in ascending order if $u_1 < u_2 < u_3 < \ldots < u_n < \ldots$
- **non-decreasing** if $u_1 \le u_2 \le u_3 \le \ldots \le u_n \le \ldots$
- **decreasing** or in descending order if $u_1 > u_2 > u_3 > \ldots > u_n > \ldots$
- **non-increasing** $u_1 \ge u_2 \ge u_3 \ge \ldots \ge u_n \ge \ldots$

A sequence that is either non-decreasing or non-increasing is called **monotone**, and a sequence that is increasing or decreasing is called **strictly monotone.**

Teaching guidelines

Let learners know sets of numbers. You can request them to write down set of even numbers and set of odd numbers (arithmetic sequences).

- Organise the class into groups. Request each group to have a group leader who will present their findings to the class.
- Request each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work. Let learners interact through questions and comments.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

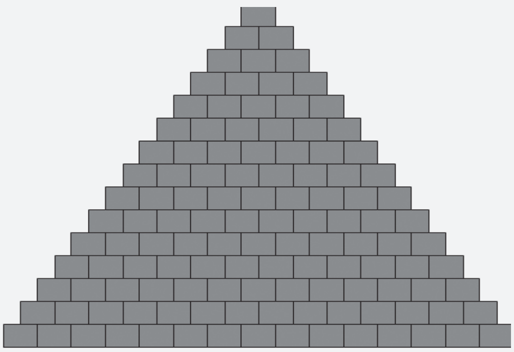
Activity 2.1 Page 40

Materials

Exercise book, pens, pencil, ruler, calculator

Answers

1. a)



   b) There are 15 rows

   c) 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1

   d) That number is -1

2. Number of insects in

1^{st} generation $\to 126$, 2^{nd} generation $\to 252$,
3^{rd} generation $\to 504$, 4^{th} generation $\to 1008$,
5^{th} generation $\to 2016$, 6^{th} generation $\to 4032$,
7^{th} generation $\to 8064$, 8^{th} generation $\to 16128$,
9^{th} generation $\to 32256$, 10^{th} generation $\to 64512$

Exercise 2.1 Page 44

1. $1, \frac{8}{5}, \frac{18}{10}$

2. $\sqrt{2}, \sqrt{3}-\sqrt{2}, 2-\sqrt{3}, \sqrt{5}-2, \sqrt{6}-\sqrt{5}$

3. $\{2n-1\}_{n=1}^{+\infty}$

Activity 2.2 Page 44

Materials

Exercise book, pens

Answers

1. 0	2. 0	3. $+\infty$

Exercise 2.2 Page 45

1. Converges to 2	2. Converge to -1
3. Converges to -5	4. Diverge
5. Converges to $\frac{2}{\sqrt{3}}$	6. Converges to 0

Activity 2.3 Page 46

Materials

Exercise book, pens

Answers

1. Ascending	2. Descending
3. Both	4. Neither

Exercise 2.3 Page 48

1. Increasing	2. Increasing
3. Decreasing	4. Both increasing, decreasing
5. Not monotonic	

2.2. Arithmetic sequences and harmonic sequences

Recommended teaching periods: 7 periods

This section studies the arithmetic sequences and the harmonic sequence

Arithmetic progression

A finite or infinite sequence $a_1, a_2, a_3, \dots, a_n$ or $a_1, a_2, a_3, \dots, a_n, \dots$ is said to be an Arithmetic Progression (A.P.) or Arithmetic Sequence if $a_k - a_{k-1} = d$, where d is a constant independent of k, for $k = 2, 3, \dots, n$ or $k = 2, 3, \dots, n, \dots$ as the case may be.

Characteristics

If three consecutive terms, u_{n-1}, u_n, u_{n+1} are in arithmetic sequence, then, $2u_n = u_{n-1} + u_{n+1}$

- Common difference

  In A.P., the difference between any two consecutive terms is a constant d, called **common difference**

- **General term or nth term**

  The nth term, u_n, of an arithmetic sequence $\{u_n\}$ with common difference d and initial term u_1 is given by

  $u_n = u_1 + (n-1)d$

  Generally, if u_p is any p^{th} term of a sequence, then the nth term is given by $u_n = u_p + (n-p)d$

- **Arithmetic means**

  If three or more than three numbers are in arithmetic sequence, then all terms lying between the first and the last numbers are called arithmetic means. If B is arithmetic mean between A and C, then $B = \frac{A+C}{2}$.

  To insert k arithmetic means between two terms u_1 and u_n is to form an arithmetic sequence of $n = k+2$ terms whose first term is u_1 and the last term is u_n.

- **Sum of first n terms or arithmetic series**

  The sum of first n terms of a finite arithmetic sequence with initial term u_1 is given by $S_n = \frac{n}{2}(u_1 + u_n)$ which is called finite arithmetic series

Harmonic sequence

A sequence is said to be in harmonic progression if the reciprocals of its terms form an arithmetic progression.

Characteristics

If three consecutive terms, h_{n-1}, h_n, h_{n+1} are in arithmetic sequence, then, $\frac{2}{h_n} = \frac{h_{n-1} + h_{n+1}}{h_{n-1}\, h_{n+1}}$ or

$$\frac{h_n}{2} = \frac{h_{n-1}h_{n+1}}{h_{n-1} + h_{n+1}} \Leftrightarrow h_n = \frac{2h_{n-1}h_{n+1}}{h_{n-1} + h_{n+1}}$$

General term or nth term of H.P.

Take the reciprocals of the terms of the given series; these reciprocals will be in A.P.

Find n^{th} term of this A.P. using $u_n = u_1 + (n-1)d$ or $u_n = u_p + (n-p)d$

Take the reciprocal of the n^{th} term of A.P., to get the required n^{th} term of H.P.

Thus, the n^{th} term of H.P. is $\frac{1}{u_1 + (n-1)d}$ or $\frac{1}{u_p + (n-p)d}$

- **Harmonic means**

  If three or more than three numbers are in harmonic sequence, then all terms lying between the first and the last numbers are called harmonic means. If B is harmonic mean between A and C, then $B = \frac{A+C}{2}$.

  To insert k harmonic means between two terms h_1 and h_n is to form a harmonic sequence of $n = k+2$ terms whose first term is h_1 and the last term is h_n.

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 2.4 Page 49

Materials

Exercise book, pens, calculator

Answers

1. 3
2. -6

Exercise 2.4 Page 52

1. $x = 9$
2. No. The number added to the 4th term to obtain the 5th term is not equal to the one used for previous first terms.
3. Common difference is 2
4. $x = 3 \; or \; 7$, fourth term: 0 or 60

Activity 2.5 Page 52

Materials

Exercise book, pens

Answers

$$u_2 = u_1 + d$$

$$u_3 = u_2 + d = \left(u_1 + d\right) + d = u_1 + d$$

$$u_4 = u_3 + d = \left(u_1 + 2d\right) + d = u_1 + 3d$$

$$u_5 = u_4 + d = \left(u_1 + 3d\right) + d = u_1 + 4d$$

$$u_6 = u_5 + d = \left(u_1 + 4d\right) + d = u_1 + 5d$$

$$u_7 = u_6 + d = \left(u_1 + 5d\right) + d = u_1 + 6d$$

$$u_8 = u_7 + d = \left(u_1 + 6d\right) + d = u_1 + 7d$$

$$u_9 = u_8 + d = \left(u_1 + 7d\right) + d = u_1 + 8d$$

$$u_{10} = u_9 + d = \left(u_1 + 8d\right) + d = u_1 + 9d$$

Generally,

$$u_n = u_{n-1} + d = \left(u_1 + (n-2)d\right) + d = u_1 + \left(n-1\right)d$$

Exercise 2.5 Page 54

1. 3	2. 9	4. 1
5. 336metres	6. None of these answers	

Activity 2.6 Page 55

Materials

Exercise book, pens, calculator

Answers

$u_1 = 2, u_7 = 20$ $\qquad$ $u_n = u_1 + \left(n-1\right)d \Rightarrow u_7 = u_1 + 6d$

$\Rightarrow 20 = 2 + 6d$ $\qquad$ $\Rightarrow d = 3$

The sequence is $2, 5, 8, 11, 14, 17, 20$

Exercise 2.6 Page 56

1. $-3,-1,1,3,5,7$	2. $2,5,8,11,14,17,20,23,26,29,32$
3. 16	4. 14
5. 0	

Activity 2.7 Page 57

Materials

Exercise book, piece of paper (or manila paper), pens

Answers

$$u_1 = u_1$$
$$u_2 = u_1 + d$$
$$\vdots$$
$$u_{n-1} = u_1 + (n-2)d$$
$$u_n = u_1 + (n-1)d$$

Let s_n denote the sum of these terms.

We have

$$[u_1 + d]\ s_n = u_1 + [u_1 + d] + \ldots + [u_1 + (n-2)d] + [u_1 + (n-1)d]$$

Reversing the order of the sum, we obtain

$$s_n = [u_1 + (n-1)d] + [u_1 + (n-2)d] + \ldots + [u_1 + d] + u_1$$

Adding the left sides of these two equations and corresponding elements of the right sides,

we see that:

$$2s_n = [2u_1 + (n-1)d] + [2u_1 + (n-1)d] + \ldots + [2u_1 + (n-1)d]$$
$$= n[2u_1 + (n-1)d]$$

$$\Leftrightarrow s_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}[u_1 + u_1 + (n-1)d]$$

By replacing $u_1 + (n-1)d$ with u_n, we obtain a useful formula for the sum:

$$s_n = \frac{n}{2}[u_1 + u_n]$$

or $s_n = \frac{n}{2}(u_1 + u_1(n-1)d) \Rightarrow s_n = \frac{n}{2}(2u_1 + (n-1)d)$

Exercise 2.7 Page 58

1. $2n(n+3)$ 2. 860 3. 11

Activity 2.8 Page 59

Materials

Exercise book, piece of paper (or manila paper), pens

Answers

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}$

The denominators are in arithmetic progression.

Exercise 2.8 Page 62

1. The sequence is $6, 4, 3, \frac{12}{5}, 2, \frac{12}{7}, \frac{3}{2}, \frac{4}{3}$. 4^{th} *term is* $\frac{12}{5}$, 8^{th} *term is* $\frac{4}{3}$

2. $3, \frac{90}{23}, \frac{90}{16}, 10$

3. $\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{4}, \frac{1}{\sqrt{5}}, \ldots, \frac{\sqrt{5}}{13}$

4. 6 and 2

5. $4 + 4\frac{2}{7} + 4\frac{8}{13} + 5 + \cdots$

6. $\frac{60}{16-n}$

2.3. Geometric sequences

Recommended teaching periods: 7 periods

This section studies the geometric sequences

- Definition

  A Geometric Progression (G.P.) or Geometric Sequence is a sequence in which each term is a fixed multiple of the previous term i.e. $\frac{a_k}{a_{k-1}} = r$, where r is a constant independent of k, for $k = 2, 3, \ldots, n$ or $k = 2, 3, \ldots, n, \ldots$.

- Characteristics

  If three consecutive terms, u_{n-1}, u_n, u_{n+1} are terms in geometric progression, then, $u_n^2 = u_{n-1}\, u_{n+1}$

- Common ratio

  In G.P., the ratio between any two consecutive terms is a constant r, called **common ratio**

- General term or n^th term

  The n^th term, u_n, of a geometric sequence $\{u_n\}$ with common ratio r and initial term u_1 is given by $u_n = u_1 r^{n-1}$

  Generally, if u_p is any p^{th} term of a sequence, then the n^th term is given by $u_n = u_p r^{n-p}$

Geometric means

If three or more than three numbers are in geometric sequence, then all terms lying between the first and the last numbers are called geometric means.

To insert *k* geometric **means** between two terms u_1 and u_n is to form a geometric sequence of $n = k + 2$ terms whose first term is u_1 and the last term is u_n.

- Sum of first n[th] terms or geometric series

  The sum of first *n* terms of a finite geometric sequence with initial term u_1 is given by $S_n = \frac{u_1(1-r^n)}{1-r}, r<1$ or $S_n = \frac{u_1(r^n-1)}{r-1}, r>1$ which is called **finite geometric series**

  If the initial term is u_0, then the formula is $s_n = \frac{u_0(1-r^{n+1})}{1-r}$ with $r \neq 1$

  If $r=1$, $s_n = nu_1$

  Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

  If the initial term is u_0 then $P_n = (u_0)^{n+1} r^{\frac{n}{2}(n+1)}$

Teaching guidelines

Let learners know what arithmetic sequence is. Recall that for an arithmetic sequence, we add a constant number to the term to obtain the next term.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understandthe activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 2.9 Page 62

Materials

Exercise book, piece of paper (or manila paper), pens, calculator, scissors or blades

Answers

Learners will take a piece of paper and cut it into two equal parts. Take one part and cut it again into two equal parts. When they continue in this manner the fraction corresponding to the obtained parts according to the original piece of paper are as follows:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Exercise 2.9 Page 64

1. $x = -6 \ or \ 6$
2. No, the number multiplied to the fourth term to obtain fifth term is not the same as the one used for previous terms.
3. Common ratio is -2
4. $k = -\frac{1}{2}$

Activity 2.10 Page 65

Materials

Exercise book, pens

Answers

$u_2 = u_1 r$ $\quad$ $u_3 = u_2 r = u_1 r r = u_1 r^2$

$u_4 = u_3 r = u_1 r^2 r = u_1 r^3$ $\quad$ $u_5 = u_4 r = u_1 r^3 r = u_1 r^4$

$u_6 = u_5 r = u_1 r^4 r = u_1 r^5$ $\quad$ $u_7 = u_6 r = u_1 r^5 r = u_1 r^6$

$u_8 = u_7 r = u_1 r^6 r = u_1 r^7$ $\quad$ $u_9 = u_8 r = u_1 r^7 r = u_1 r^8$

$u_{10} = u_9 r = u_1 r^8 r = u_1 r^9$

Generally,

$u_n = u_{n-1} r = u_1 r^{n-2} r = u_1 r^{n-1}$

Exercise 2.10 Page 67

1. 98304
2. $\dfrac{\sqrt[5]{16}}{4}$
3. -21.87
4. $\dfrac{1}{16}$
5. $(u_n): u_n = \dfrac{1}{2}\left(\dfrac{3}{2}\right)^{n-1}$, $u_8 = \dfrac{2187}{256}$
6. $p = 5$

Activity 2.11 Page 67

Materials

Exercise book, pens, calculator

Answers

$u_1 = 1, u_6 = 243$ $\quad$ $u_n = u_1 \cdot r^{n-1} \Rightarrow u_6 = u_1 \cdot r^5$

$\Rightarrow 243 = r^5$ $\quad$ $\Rightarrow 3^5 = r^5$

$\Rightarrow r = 3$

The sequence is $1, 3, 9, 27, 81, 243$

Exercise 2.11 Page 69

1. $\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64},\frac{1}{128},\frac{1}{256}$ or $\frac{1}{4},-\frac{1}{8},\frac{1}{16},-\frac{1}{32},\frac{1}{64},-\frac{1}{128},\frac{1}{256}$
2. $2,\frac{2}{3},\frac{2}{9},\frac{2}{27},\frac{2}{81},\frac{2}{243},\frac{2}{729}$ or $2,-\frac{2}{3},\frac{2}{9},-\frac{2}{27},\frac{2}{81},-\frac{2}{243},\frac{2}{729}$
3. a) 14 b) $\frac{9}{2}$
4. 12 and 108
5. 64 and 4

Activity 2.12 Page 69

Materials

Exercise book, pens

Answers

1. Let $s_n = u_1 + u_2 + u_3 + ... + u_n$

   $s_n = u_1 + u_1 r + u_1 r^2 + ... + u_1 r^{n-1}$ (1)

   Multiply both sides of (1) by r, we obtain

   $s_n r = u_1 r + u_1 r^2 + u_1 r^3 + ... + u_1 r^n$ (2)

   Subtract (2) from (1), we get

   $$\begin{array}{l} s_n = u_1 + u_1 r + u_1 r^2 + ... + u_1 r^{n-1} \\ \underline{-s_n r = \quad -u_1 r - u_1 r^2 - u_1 r^3 - ... - u_1 r^n} \\ s_n - s_n r = u_1 - u_1 r^n \end{array}$$

   $\Leftrightarrow s_n(1-r) = u_1(1-r^n)$ or

   $s_n = \frac{u_1(1-r^n)}{1-r}$ with $r \neq 1$

2. $P = u_1 \times u_1 r \times u_1 r^2 \times ... \times u_1 r^{n-1}$

   $\Leftrightarrow P = (u_1)^n (r \times r^2 \times ... \times r^{n-1})$

   $\Leftrightarrow P = (u_1)^n r^{(1+2+...+n-1)}$

We need the sum $S_{n-1} = 1+2+...+n-1$

$$S_{n-1} = \frac{n-1}{2}(1+n-1) = \frac{n(n-1)}{2}$$

Then $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

Exercise 2.12 Page 72

1) 21.25 2) 39.1 3) $1, \frac{5}{4}$ 4) -32

Activity 2.13 Page 72

Materials

Exercise book, pens

Answers

If $-1 < r < 1$, $\lim_{n\to\infty} r^n = 0$

thus $\lim_{n\to\infty} \frac{u_1(1-r^n)}{1-r} = \frac{u_1}{1-r}$.

Exercise 2.13 Page 73

1. a) $0 < x < \frac{4}{3}$ b) $-\frac{190}{39}$
2. $115m$

2.4. Applications

Recommended teaching periods: 4 periods

This section studies the applications of sequences in daily life.

Teaching guidelines

Let learners now solve any problem related to sequences. Request them to read books and find out how sequences can be used in real life problems and request them to present their findings.

Activity 2.14 Page 73

Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on how sequences are used in real life.

For example; the interest portion of monthly payments made to pay off an automobile or home loan, and the list of maximum daily temperatures in one area for a month are sequences. One application in economy is calculation of interest rate. The compound interest formula: $A = P\left(1+\frac{r}{k}\right)^{kt}$ with P = principle, t = time in years, r = annual rate, and k = number of periods per year. The simple interest formula: $I = \mathrm{P}rt$ with I = total interest, P = principle, r = annual rate, and t = time in years.

End of Unit Assessment Page 79

1. a) $0,-\frac{1}{4},-\frac{2}{9},-\frac{3}{16}$ b) $1,-\frac{1}{3},\frac{1}{5},-\frac{1}{7}$ c) $1,3,1,3$
2. a) $(-1)^n,\ n=0,1,2,..$ b) $n^2-1,\ n=1,2,3,...$
   c) $4n-3,\ n=1,2,3,...$
3. a) converges to $\sqrt{2}$ b) converges to 0
   c) converges to 1
4. a) $u_{20}=78, S_{20}=800$ b) $u_{20}=23.5, S_{20}=185$
5. a) $u_n=2(n+1), S_n=n(n+3)$
   b) $u_n=20-3n, S_n=\frac{n}{2}(37-3n)$
   c) $u_n=\frac{1}{n}, S_n=\frac{n+1}{2}$
6. a) $u_8=18, S_8=88$ b) $u_1=3, S_{10}=210$
   c) $n=10, d=2$ d) $u_1=1, d=2$

7. $\frac{157}{4},\frac{79}{2},\frac{159}{4},40,\frac{161}{4},\frac{81}{2},\frac{163}{4},41,\frac{165}{4},\frac{83}{2},\frac{167}{4},42,...$

8. $5,7,9,11,13,15,17,19,21,23,25,27,29,31,...$

9. $-2,-\frac{7}{4},-\frac{6}{4},-\frac{5}{4},-1,-\frac{3}{4},-\frac{2}{4},-\frac{1}{4},0,\frac{1}{4}$

10. 8,9,10

11. 9,11,13 or 13,11,9

12. $\frac{6}{n}$

13. $1,\frac{2}{5},\frac{1}{4}$

14. $\frac{5}{37}$

15. a) $u_5=768, S_5=341$ b) $r=\frac{1}{2}, S_n=\frac{1533}{64}$

16. $\frac{2}{2b}=\frac{1}{b-a}+\frac{1}{b-c} \Leftrightarrow \frac{1}{b}=\frac{2b-a-c}{b^2-bc-ab+ac}$

$\Leftrightarrow b^2+ac=2b^2 \Rightarrow b^2=ac$. This shows that a,b,c form a geometric progression.

17. $u_5=\frac{81}{2}$

18. $u_8=\frac{2187}{4}$

19. $2,2\sqrt{2},4,4\sqrt{2},8$ or $2,-2\sqrt{2},4,-4\sqrt{2},8$

20. 3,6,12

21. 128 or -972

22. 6, 6, 6 or 6, -3, -12.

23. 11,17,23

24. 5,8,11,14

25. -4,-1,2,5,8

26. 2,3

27. 6

28. £11 million

29. —

30. $\frac{4}{5},5$

31. 2048000

32. $99.8^0 F$

33. 1800

34. 6 and 3

35. $\frac{2(a+b)\sqrt{ab}}{\left(\sqrt{a}+\sqrt{b}\right)^2}$

36. $a=b=c$

Unit 3 Logarithmic and Exponential Equations

Learner's Book page 83 – 104

Aim

Solve equations involving logarithms or exponentials and apply them to model and solve related problems

Objectives

- Solve simple exponential equations.
- Convert a number from logarithmic form to exponential form.
- Change the base of any logarithm.
- Use the properties of logarithms to solve logarithmic and exponential equations.
- Apply logarithms or exponential to solve interest rate problems, population growth problems, radioactivity decay problems, earthquake problems,…

Materials

Exercise books, pens, instruments of geometry, calculator

Contents

3.1. Exponential and logarithmic functions

Recommended teaching periods: 7 periods

This section looks at how to sketch exponential and logarithmic function in Cartesian plane.

The graph of a logarithmic function is found by reflecting the graph of the corresponding exponential function to the **first bisector** i.e. the line $y = x$.

Then the coordinates of the points for $y = a^x$ are reversed to obtain the coordinates of the points for $g(x) = \log_a x$.

Teaching guidelines

Let learners know how to draw linear function in 2-dimensions.

Recall that to sketch a function you need a table of points.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in the Learner's Book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

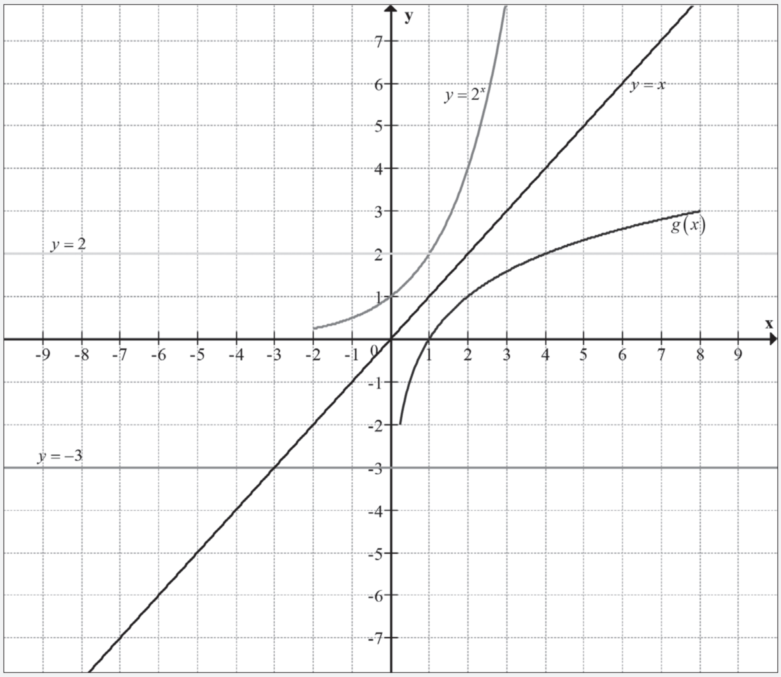
Answers to activities and exercises

Activity 3.1 Page 84

Materials

Exercise book, pens, instruments of geometry, calculator

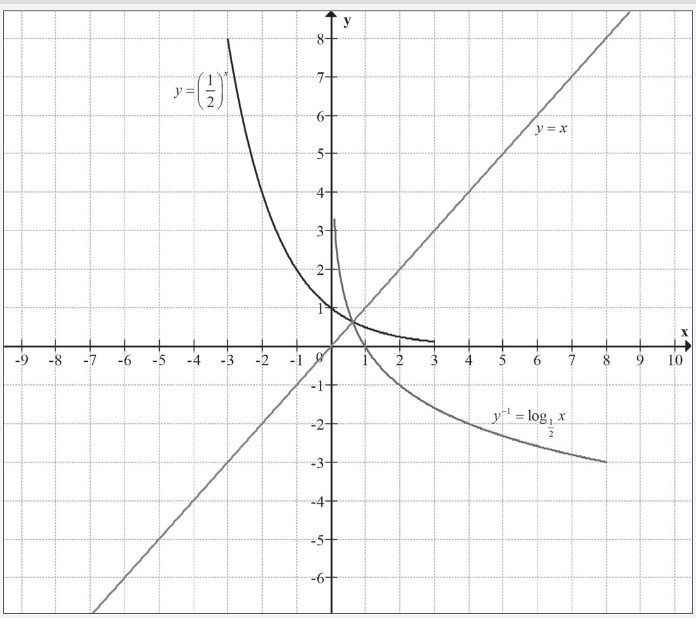
Answers

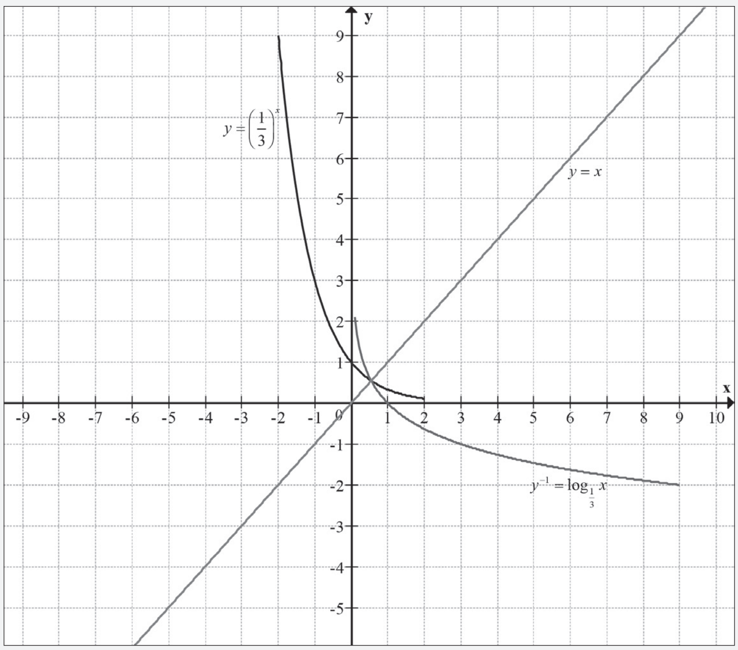


Any horizontal line crosses the curve at most once

$y = 2^x$ is one to one function (is invertible function)

Exercise 3.1 Page 85





3.2. Exponential and logarithmic equations

Recommended teaching periods: 7 periods

This section looks at the method used to solve exponential and logarithmic equations.

In solving exponential or logarithmic equations, remember basic rules for exponents and/or logarithms.

Basic rules for exponents

For $a>0$ *and* $a\neq 1, m, n\in IR$

a) $a^m\times a^n=a^{m+n}$

b) $a^m : a^n = a^{m-n}$

c) $\left(a^m\right)^n=a^{mn}$

d) $a^{-n}=\dfrac{1}{a^n}$

e) $a^{\frac{1}{n}}=\sqrt[n]{a}$

f) $a^{\frac{m}{n}}=\sqrt[n]{a^m}$

g) $a^{\log_a b}=b$

Basic rules for logarithms

$\forall x, y\in \left]0,+\infty\right[, a\in \left]0,+\infty\right[\setminus\{1\}:$

a) $\log_a xy=\log_a x+\log_a y$

b) $\log_a \dfrac{1}{y}=-\log_a y$

c) $\log_a \frac{x}{y} = \log_a x - \log_a y$ d) $\log_a x^r = r\log_a x \ , \forall r \in \mathbb{R}$

e) $\log_a b = \frac{\log_c b}{\log_c a} \ , \forall c \in \left]0,+\infty\right[\setminus \{1\}, \ b > 0$

Teaching guidelines

Let learners know how to solve linear and quadratic equations. They should also know basic properties of powers. Help them to recall basic properties of powers.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 3.2 Page 86

Materials

Exercise book, pens

Answers

If $m=\log_a x, n=\log_a y$ and $z=\log_a xy$ then $x=a^m$, $y=a^n$ and $xy=a^z$.

Now, $xy=a^m a^n=a^{m+n}=a^z \Rightarrow z=m+n$.

Thus, $\log_a(xy)=\log_a x+\log_a y$.

If $m=\log_a x, n=\log_a y$ and $z=\log_a \frac{x}{y}$ then $x=a^m$, $y=a^n$ and $\frac{x}{y}=a^z$.

Now, $\frac{x}{y}=\frac{a^m}{a^n}=a^{m-n}=a^z \Rightarrow z=m-n$.

Thus, $\log_a(xy)=\log_a x-\log_a y$.

Exercise 3.2 Page 88

2. a) $\log_4 64=3$ b) $\log_2 \frac{1}{8}=-3$ c) $\log_{\frac{1}{2}} y=x$

   d) $\log_p q=3$ e) $\log_8 0.5=x$ f) $\log_5 q=p$

3. a) $\log_2 8=x \Leftrightarrow 8=2^x \Leftrightarrow 2^x=2^3 \Rightarrow x=3$

   b) $\log_x 125=3 \Leftrightarrow 125=x^3 \Rightarrow x=\sqrt[3]{125} \Leftrightarrow x=5$

   c) $\log_x 64=0.5 \Leftrightarrow 64=x^{0.5} \Rightarrow x=64^2 \Leftrightarrow x=4096$

   d) $\log_4 64=x \Leftrightarrow 64=4^x \Leftrightarrow 4^x=4^3 \Rightarrow x=3$

   e) $\log_9 x=3\frac{1}{2} \Leftrightarrow x=9^{3\frac{1}{2}} \Leftrightarrow x=\left(9^{\frac{1}{2}}\right)^7 \Rightarrow x=2187$

   f) $\log_2 \frac{1}{2}=x \Leftrightarrow \frac{1}{2}=2^x \Leftrightarrow 2^x=2^{-1} \Rightarrow x=-1$

4. a) 5 b) 1.5 c) -3

   d) -3 e) $\frac{1}{3}$ f) 1

   g) 1 h) 0

Activity 3.3 Page 88

Materials

Exercise book, pens

Answers

1. If $x = \log_a m$ and $z = \log_a(m^p)$ then $m = a^x$, $m^p = a^z$.

   Now, $m^p = (a^x)^p = a^{px} = a^z \Rightarrow z = px$.

   Thus, $\log_a(m^p) = p\log_a m$, as required.

2. $\log_a b = \dfrac{\log_c b}{\log_c a}$

   Let $x = \log_a b$, then $a^x = b$.

   Take logarithms in base c of both sides $\log_c a^x = \log_c b$.

   This gives $x\log_c a = \log_c b \Rightarrow x = \dfrac{\log_c b}{\log_c a}$.

   Therefore, $\log_a b = \dfrac{\log_c b}{\log_c a}$.

Exercise 3.3 Page 91

1. a) $4p$ b) $-2p$ c) $1+p$

2. $(2k\pi+1)\dfrac{\pi}{2},\ k \in \mathbb{Z}$

3. a) $\left\{\dfrac{3}{2}\right\}$ b) $\{\sqrt{2}, 2\}$ c) $\left\{\left(e^2, \dfrac{1}{e^2}\right)\right\}$

   d) $\{(e^4, e^3), (-e^4, -e^3)\}$ e) $\{2\}$

3.3. Applications

Activity 3.4 Page 92

Materials

Exercise book, pens, calculator

Answers

$P(t) = P_0 2^{kt}$

Here $P_0 = 2, k = 2, t\ in\ hours \Rightarrow P(t) = 2^{2t+1}$

a) $P(4) = 2^9 = 512$

b) $P(t) = 2^{13} \Leftrightarrow 2^{2t+1} = 2^{13} \Rightarrow 2t+1 = 13 \Rightarrow t = 6$

c) Number of cells left is $\frac{2^{22}}{2}$ $or\ 2^{21}$

Activity 3.5 Page 93

Materials

Exercise book, pens, calculator

Answers

a) The original amount of material present is

$A(0) = 80\left(2^0\right) = 80gram$

b) For the half life, $A(t) = 40$

$$40 = 80\left(2^{-\frac{t}{100}}\right) \Rightarrow \frac{1}{2} = 2^{-\frac{t}{100}} \Rightarrow 2^{-1} = 2^{-\frac{t}{100}}$$

$$1 = \frac{t}{100} \Rightarrow t = 100$$

Therefore the half life is 100 years

c) $A(t) = 1$

$$\Rightarrow 80\left(2^{-\frac{t}{100}}\right) = 1 \Rightarrow 2^{-\frac{t}{100}} = \frac{1}{80} \Rightarrow 2^{-\frac{t}{100}} = 80^{-1}$$

$$\Rightarrow \log 2^{-\frac{t}{100}} = \log 80^{-1} \Rightarrow -\frac{t}{100}\log 2 = -\log 80$$

$$\Rightarrow t = \frac{\log 80}{\log 2} \times 100 = 632$$

Therefore, it will take 632 years for material to decay to 1 gram.

Activity 3.6 Page 94

Materials

Exercise book, pens

Answers

Suppose $P(t)$ has an exponential decay model so that $P(t)=P_0e^{-kt}\quad(k<0)$.

At any fixed time t_1 let $P_1=P_0e^{-kt_1}$

be the value of $P(t)$ and let T denote the amount of time required to reduce in value by half. Thus, at time t_1+T the value of $P(t)$ will be $2P_1$ so that $2P_1=P_0e^{-k(t_1+T)}=P_0e^{-kt_1}e^{-kT}$.

Since $P_1=P_0e^{-kt_1}$,

$$2P_1=P_0e^{-k(t_1+T)}=P_0e^{-kt_1}e^{-kT}\xrightarrow{P_1=e^{-kt_1}}2P_0e^{-kt_1}=P_0e^{-kt_1}e^{-kT}$$

Or $2=e^{-kT}$, taking $\ln$ on both sides gives

$\ln 2=-kT$ or $T=-\frac{1}{k}\ln 2$ which does not depend on P_0 or t_1.

Exercise 3.4 Page 99

1. 12.5647h
2. 160.85 years
3. 866 years
4. a) a little over 95.98 b) about 66.36
5. 4,139g
6. Frw 7,557.84
7. About 14.7 years
8. 2.8147498 X 10^{14}
9. a) 10 years b) (i) 8 years (ii) 32.02 years.

End of Unit Assessment Page 101

1.	a) $\{81\}$	b) $\{-1,6\}$	c) $\{(9,7),(7,9)\}$
	d) $\left\{\frac{1}{5},5\right\}$	e) $\{2\}$	f) $\{(\ln 2,\ln 3)\}$
2.	a) 5	b) 1.5	c) 1.09
	d) 1.5	e) -3	f) 1.05
3.	a) $\$2519.42$	b) 9 years	
4.	a) 976	b) 20	
5.	12.9^0C		
6.	a) 49.7million	b) 67.1million	c) 122.4million
7.	a) 16.0	b) 28.7	c) 33.6
8.	a) 20.8years	b) 138years	
9.	$P=36.4e^{0.01t}$		
10.	a) 14,400years	b) 38years	
11.	a) 69.1^0C	b) 60.2	c) $44.4\ ^0C$
12.	a) 5.0	b) 16.2	c) 26.4
13.	c) 96min		
14.	c) After $76/77\ years$		

Unit 4 Solving Equations by Numerical Methods

Learner's Book page 105 – 126

Aim

To be able to use numerical methods (e.g Newton-Raphson method to approximate solution to equations)

Objectives

- Finding root by linear interpolation and extrapolation.
- Locating roots by graphical and analytical methods.
- Finding real root by Newton-Raphson method and general iterations.

Materials

Exercise books, pens, instruments of geometry, calculator

Contents

4.1. Linear interpolation and extrapolation

Recommended teaching periods: 4 periods

This section looks at how to use linear interpolation and extrapolation.

Linear interpolation is a process whereby the non tabulated values of the function are estimated on the assumption that the function behaves sufficiently smooth between the tabular point.

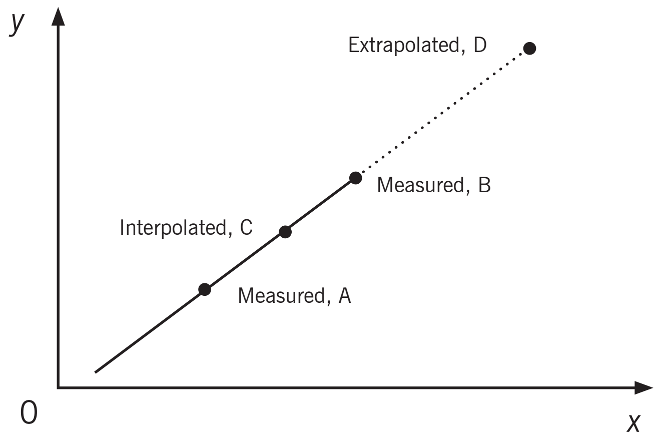
Extrapolation involves approximating the value of a function for a given value outside the given tabulated values.

The linear interpolation formula is given as

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{x_2 - x_1}$$

For extrapolation formula, we may also use the above formula .

An example of a linear interpolation is given in the graph shown below. Here, the line segment AB is given. The point C is interpolated; while the point D is extrapolated by extending the straight line beyond AB.



Teaching guidelines

Let learners know how to find equation of a line passing through given two points.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

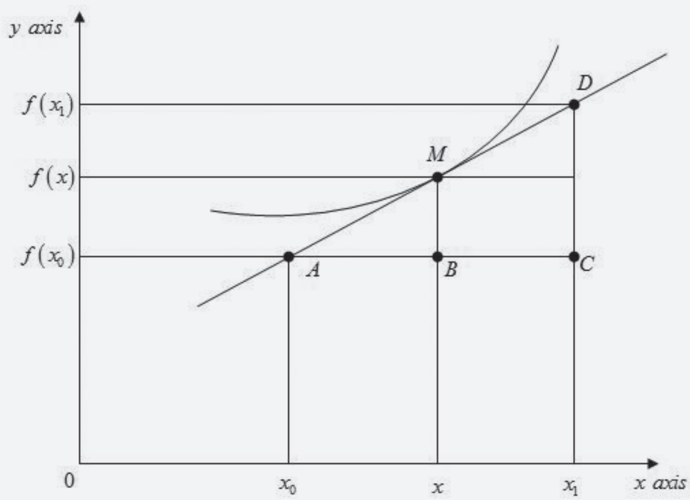
Answers to activities and exercises

Activity 4.1 Page 106

Materials

Exercise book, pens, geometric instruments

Answers



3) $\dfrac{CD}{AD}=\dfrac{f_2-f_1}{x_2-x_1}$ $\qquad$ $\dfrac{BM}{AB}=\dfrac{f-f_1}{x-x_1}$

4) Since $\dfrac{CD}{AD}=\dfrac{BM}{AB}$, thus

$$\frac{f_2-f_1}{x_2-x_1}=\frac{f-f_1}{x-x_1}$$

Or $f-f_1=\dfrac{f_2-f_1}{x_2-x_1}(x-x_1)$

$$f = \frac{f_2 - f_1}{x_2 - x_1}(x - x_1) + f_1 \qquad f = \frac{x - x_1}{x_2 - x_1}(f_2 - f_1) + f_1$$

Letting $\frac{x - x_1}{x_2 - x_1} = \delta$ and $f_2 - f_1 = \Delta f_1$ gives

$$f = \delta \Delta f_1 + f_1$$

Exercise 4.1 Page 109

1. a) $\theta = 64.3$ when T=18s

   b) T=22.5 when 60°C

2. a) 0.2324 b) 0.967

Activity 4.2 Page 109

Materials

Exercise book, pens, calculator

Answers

Let $y = ax + b$

$$a = \frac{10.5 - 10.9}{1988 - 1982} = \frac{-0.4}{6} = -0.067$$

$$y = -0.067x + b$$

$$10.5 = -0.067(1988) + b$$

$$\Rightarrow b = 143.7$$

Then, $y = -0.067x + 143.7$

The winning time in year 2010 is estimated to be:

$$y = -0.067(2010) + 143.7 = 9.03 \text{ sec}$$

Unfortunately, this estimate actually is not very accurate. This example demonstrates the weakness of linear extrapolation; it uses only a couple of points, instead of using all the points like the best fit line method, so it doesn't give as accurate results when the data points follow a linear pattern.

Exercise 4.2 Page 111

1) 11.5 2) 3.33

4.2. Location of roots

Recommended teaching periods: 8 periods

This section looks at the method used to locate root by analytical method and graphical method.

Analytical method

The root of $f(x)=0$ lies in interval $]a,b[$ if $f(a)f(b)<0$; in other words, $f(a)$ and $f(b)$ are of opposite sign.

Graphical method

To solve the equation $f(x)=0$, graphically, we draw the graph of $y=f(x)$ and read from it the value of x for which $f(x)=0$ i.e. the x-coordinates of the points where the curve $y=f(x)$ cuts the x-axis.

Alternatively, we would rearrange $f(x)=0$, in the form $h(x)=g(x)$, and find the x-coordinates of the points where the curves $y=h(x)$ and $y=g(x)$ intersect.

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 4.3 Page 111

Materials

Exercise book, pens, calculator

Answers

- Table of values for $y = x^2 - 5x + 2$

x	0	1	2	3	4	5
$y = x^2 - 5x + 2$	2	-2	-4	-4	-2	2

- The ranges of root of equation $x^2 - 5x + 2 = 0$:

$f(0) = 2 > 0$ and $f(1) = -2 < 0$, so, a root lies between 0 and 1

$f(4) = -2 < 0$ and $f(5) = 2 > 0$, so, a root lies between 4 and 5.

The ranges of root of equation $x^2 - 5x + 2 = 0$ are $0 < x < 1$ and $4 < x < 5$

Exercise 4.3 Page 112

1. $f(x) = x^3 - 3x - 12$

$f(2) = 2^3 - 6 - 12 = -10 < 0$

$f(3) = 3^3 - 9 - 12 = 6 > 0$

Since $f(2)f(3) < 0$, thus, the equation $x^3 - 3x - 12 = 0$ has a root between 2 and 3

x	2	?	3
y	-10	0	6

Hint:

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{x_2 - x_1} \Rightarrow x = x_1 + \frac{(y - y_1)(x_2 - x_1)}{y_2 - y_1}$$

$x = 2.625$

2. $f(x) = 3x^2 + x - 5$

$f(1) = 3 + 1 - 5 = -1 < 0$

$f(2) = 12 + 1 - 5 = 8 > 0$

As $f(1)f(2) < 0$, then, the equation $3x^2 + x - 5 = 0$ has a root between 1 and 2.

x	1	?	2
y	-1	0	9

$$x = x_1 + \frac{(y - y_1)(x_2 - x_1)}{y_2 - y_1}$$

$x = 1.1$

Activity 4.4 Page 113

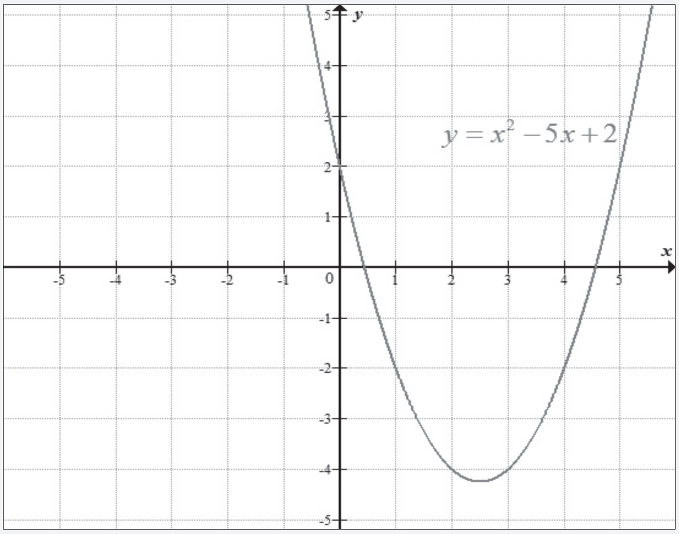
Materials

Exercise book, pens, calculator, geometrical instruments

Answers

1. Graph of $y = x^2 - 5x + 2$

x	0	1	2	3	4	5
$y = x^2 - 5x + 2$	2	-2	-4	-4	-2	2



2. Ranges of roots: $]0,1[$ and $]4,5[$

3. $x^2-5x+2=0 \Leftrightarrow x^2=5x-2$

   $\Rightarrow x=\sqrt{5x-2}$ or $x=-\sqrt{5x-2}$

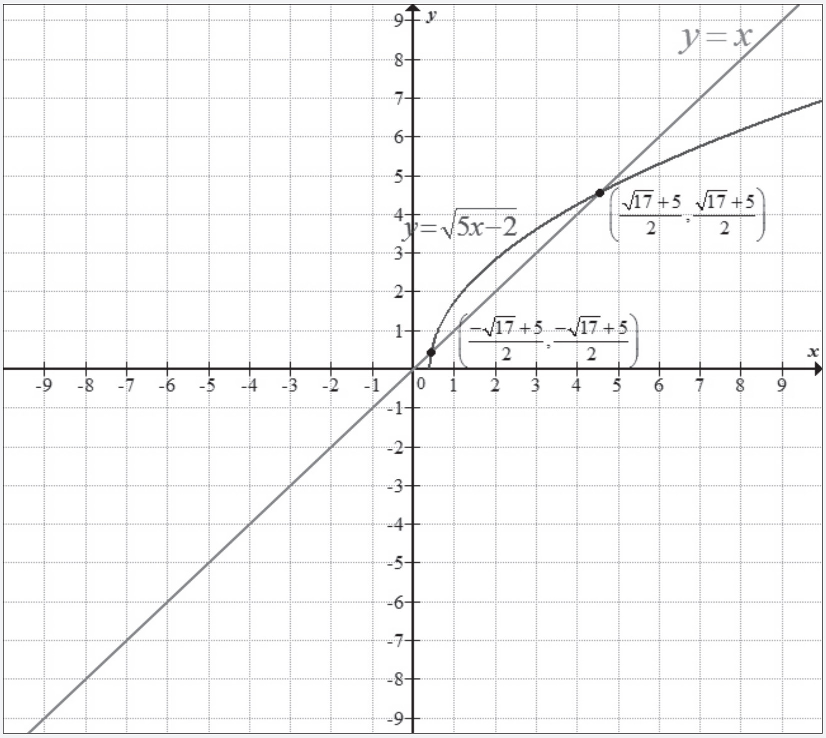
4. Taking the positive root, we get

   $y=x$ and $y=\sqrt{5x-2}$

x	0.4	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0.4	1	1.5	2	2.5	3	3.5	4	4.5	5

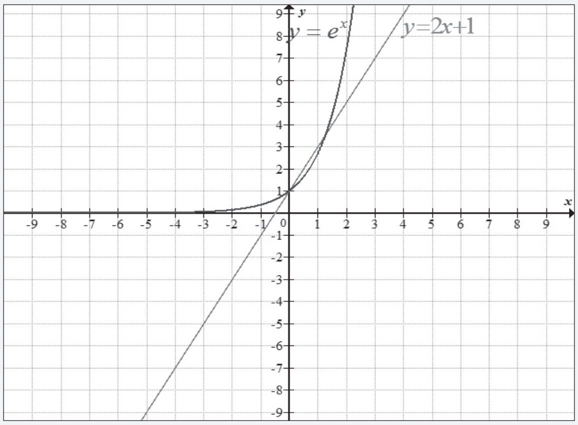
x	0.4	1	1.5	2	2.5	3	3.5	4	4.5	5
$y=\sqrt{5x-2}$	0.0	1.7	2.3	2.8	3.2	3.6	3.9	4.2	4.5	4.8

5. The graphs of $y=x$ and $y=\sqrt{5x-2}$ look like this

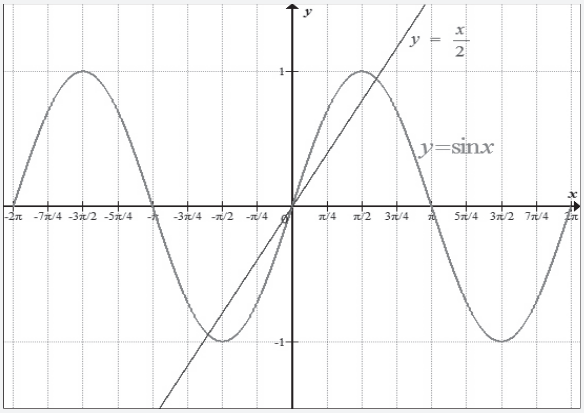


Exercise 4.4 Page 115

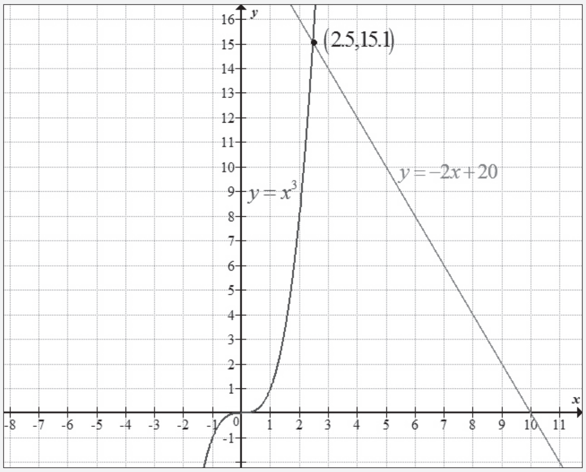
1.



2.

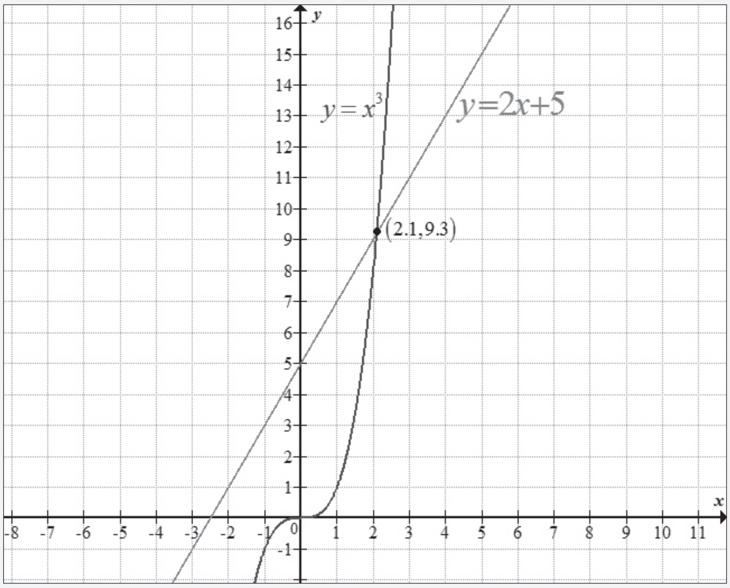


3.



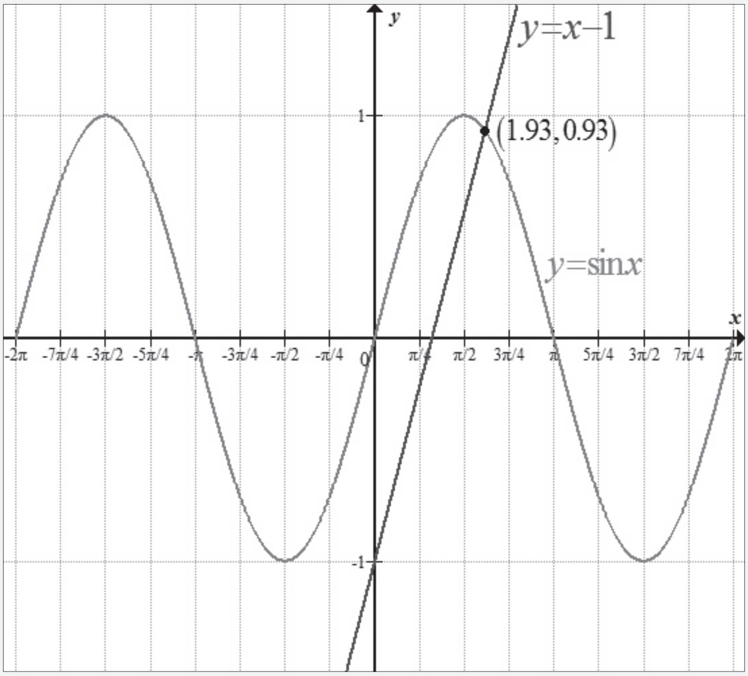
The root is 2.5

4.



The root is 2.1

5.



The root is 1.9

4.3. Iteration method

Recommended teaching periods: 9 periods

This section looks at the method used to find roots by Newton-Raphson method and general iterations.

By this method, we get closer approximation of the root of an equation if we already know its good approximate root.

1. Guess a first approximation to a solution of the equation $f(x)=0$. A graph of $y=f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on using the formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\ if\ f'(x_n)\neq 0$$

Notice

If $f'(x_1)=0$ or nearly zero, this method fails.

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

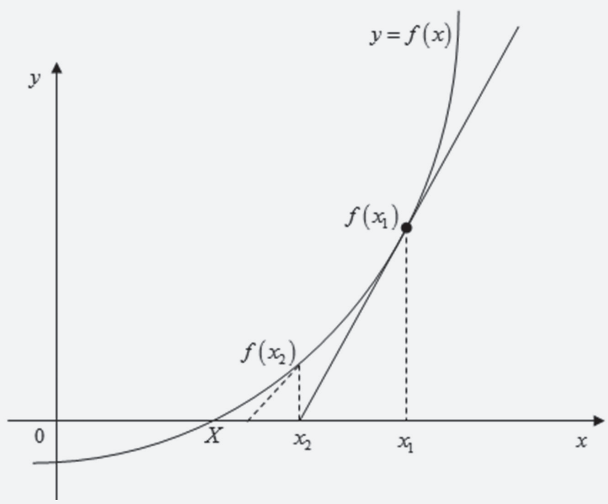
Activity 4.5 Page 116

Materials

Exercise book, pens and geometric instruments

Answers

1.



Slope of the tangent line: $m = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(x)$

Equation of tangent: $f(x) - f(x_1) = f'(x_1)(x - x_1)$

2. At $X = x$, $f(x) = 0$. The tangent equation will be $y = m$

x_2 is a best approximation.

From tangent equation, we get: $-f(x_1) = f'(x_1)(x_2 - x_1)$

$$-\frac{f(x_1)}{f'(x_1)} = x_2 - x_1$$

Or

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Exercise 4.5 Page 120

1.	1.521	2.	-2.104	3.	0.79206
4.	1.224	5.	0.581	6.	0.619

Activity 4.6 Page 121

Materials

Exercise book, pens, calculator

Answers

$$x^3 - 3x - 5 = 0 \Leftrightarrow x^3 = 3x + 5$$

$$\Rightarrow x = \sqrt[3]{3x+5}, \quad x_1 = 2$$

$$x_2 = \sqrt[3]{3\times 2+5} = \sqrt[3]{11} = 2.22398$$

$$x_3 = \sqrt[3]{3\times 2.22398+5} = 2.268372$$

$$x_4 = \sqrt[3]{3\times 2.268372+5} = 2.276967$$

$$x_5 = \sqrt[3]{3\times 2.276967+5} = 2.278624$$

$$x_6 = \sqrt[3]{3\times 2.278624+5} = 2.278943$$

$$x_7 = \sqrt[3]{3\times 2.278943+5} = 2.279004$$

$$x_8 = \sqrt[3]{3\times 2.279004+5} = 2.279016$$

$$x = 2.279 \text{ to } 3dp$$

Exercise 4.6 Page 123

1. a) $x^2 - 3x + 1 = 0$

$$f(x) = x^2 - 3x + 1$$

$$f(0) = 0^2 - 0 + 1 = 1 > 0$$

$$f(1) = 1^2 - 3 + 1 = -1 < 0$$

As $f(0)f(1) < 0$, then, the equation $x^2 - 3x + 1 = 0$ has a root between 0 and 1.

$$f(2) = 2^2 - 6 + 1 = -1 < 0$$
$$f(3) = 3^2 - 9 + 1 = 1 > 0$$

As $f(2)f(3) < 0$, then, the equation $x^2 - 3x + 1 = 0$ has a root between 2 and 3.

b) i) $x^2 - 3x + 1 = 0 \Leftrightarrow 3x = x^2 + 1$

$$\Leftrightarrow x = \frac{x^2 + 1}{3} \Rightarrow p = 1, q = 3$$

ii) $x^2 - 3x + 1 = 0 \Leftrightarrow x^2 - 3x = -1$

$$\Leftrightarrow x(x-3) = -1$$

$$\Rightarrow x - 3 = -\frac{1}{x}, x \neq 0$$

$$\Leftrightarrow x = 3 - \frac{1}{x} \Rightarrow r = 3, s = -1$$

c) $x = \underline{\qquad}$, $x = 0.5$, $x_2 = \dfrac{(0.5)^2 + 1}{3} = 0.416667$

$$x_3 = \frac{(0.416667)^2 + 1}{3} = 0.391204$$

$$x_4 = \frac{(0.391204)^2 + 1}{3} = 0.384347$$

$$x_5 = \frac{(0.384347)^2 + 1}{3} = 0.382574$$

$$x_6 = \frac{(0.382574)^2 + 1}{3} = 0.382121$$

$$x_7 = \frac{(0.382121)^2 + 1}{3} = 0.382005$$

$$x_8 = \frac{(0.382005)^2 + 1}{3} = 0.381976$$

$$x_9 = \frac{(0.381976)^2 + 1}{3} = 0.381969$$

$$x = 0.382 \ to\, 3\, dp$$

d) $$x = 3 - \frac{1}{x}, \ x_1 = 0.5, \ x_2 = 3 - \frac{1}{0.5} = 1$$

$$x_3 = 3 - \frac{1}{1} = 2$$

$$x_4 = 3 - \frac{1}{2} = 2.5$$

$$x_5 = 3 - \frac{10}{26} = 2.615385$$

$$x_6 = 3 - \frac{1}{2.615385} = 2.615385$$

$$x_7 = 3 - \frac{1}{2.615385} = 2.617647$$

$$x_8 = 3 - \frac{1}{2.617647} = 2.617978$$

$$x = 2.618 \ to\, 3\, dp$$

2. $$x_{n+1} = 2 + \frac{1}{x_n^2}, \ x_0 = 2$$

$$x_1 = 2 + \frac{1}{2^2} = 2.25$$

$$x_2 = 2 + \frac{1}{(2.25)^2} = 2.197531$$

$$x_3 = 2 + \frac{1}{(2.197531)^2} = 2.207076$$

$$x_4 = 2 + \frac{1}{(2.207076)^2} = 2.205289$$

$$x_5 = 2 + \frac{1}{(2.205289)^2} = 2.205622$$

$$x_6 = 2 + \frac{1}{(2.205622)^2} = 2.20556$$

$$x_7 = 2 + \frac{1}{(2.20556)^2} = 2.205571$$

$$x_8 = 2 + \frac{1}{(2.205571)^2} = 2.205569$$

$$x_9 = 2 + \frac{1}{(2.205569)^2} = 2.205569$$

$x = 2.205569\ to\ 6\ dp$

Equation is $x^3 - 2x^2 - 1 = 0$

3. $f(x) = x^2 - \sin x$

$f(0.5) = (0.5)^2 - \sin 0.5 = -0.22943 < 0$

$f(1) = 1^2 - \sin 1 = 0.158529 > 0$

As $f(0.5)f(1) < 0$, then, the equation $x^2 - \sin x = 0$ has a root between 0.5 and 1.

$$x_{n+1} = \frac{\sin x}{x_n},\ x_0 = 0.5$$

$$x_1 = \frac{\sin 0.5}{0.5} = 0.958851$$

$$x_2 = \frac{\sin 0.958851}{0.958851} = 0.853659$$

$$x_3 = \frac{\sin 0.853659}{0.853659} = 0.882894$$

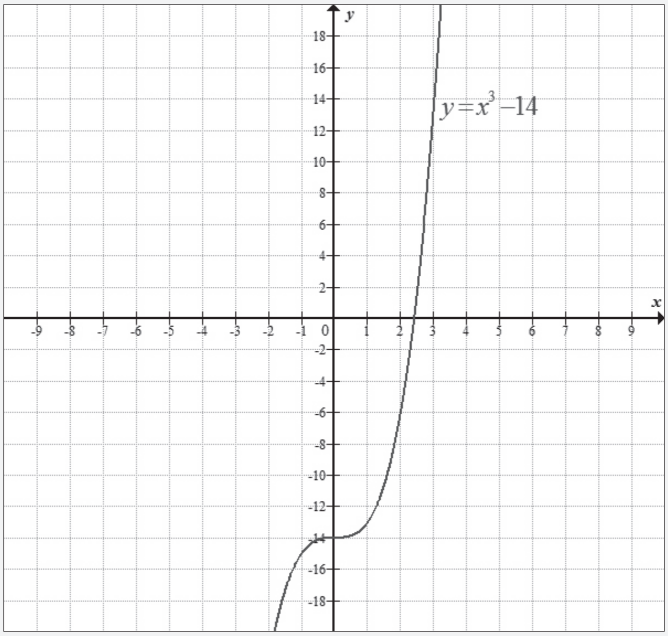
$$x_4 = \frac{\sin 0.882894}{0.882894} = 0.875054$$

$$x_5 = \frac{\sin 0.875054}{0.875054} = 0.877178$$

$x = 0.877\ to\ 3\ dp$

End of Unit Assessment Page 125

1. a)



The root lies between 2 and 3

$$x = \frac{p}{x^2} + \frac{x}{2}$$

$$\Leftrightarrow x = \frac{2p + x^3}{2x^2} \quad \Leftrightarrow 2x^3 = 2p + x^3$$

$$\Leftrightarrow x^3 = 2p$$

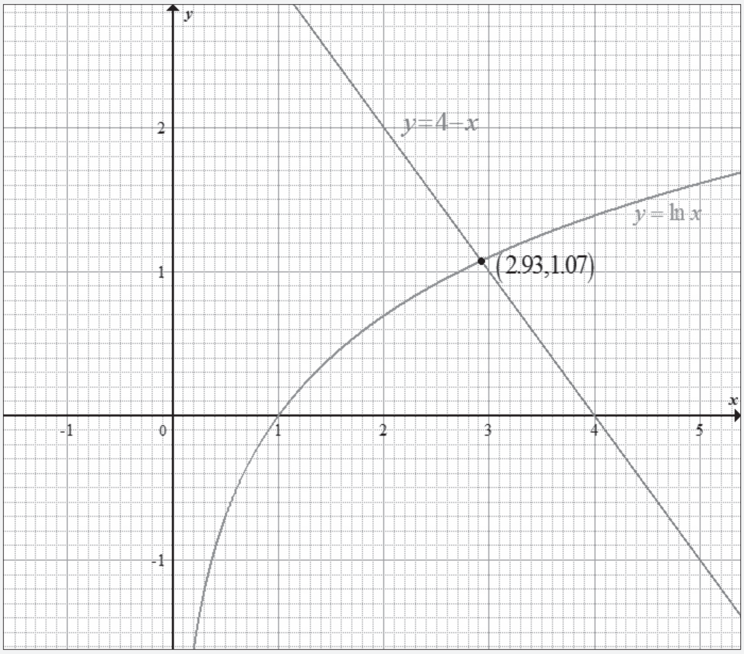
$$2p = 14 \Rightarrow p = 7$$

b) 2.410

2. 1.8171206

3. a) 1.67 b) 1.16 c) 1.9 d) 0.85

4. Graph



From the graph, the root lies between 2.9 and 3.

Answer to three significant figures is 2.93

5. 1.738

6. a) 1.4973 b) 3.332

7. 1.83, $x = 3^{\frac{1}{x}}$

8. a) 0.45 b) -4 is undefined c) -4.45

Unit 5 Trigonometric Functions and their Inverses

Learner's Book page 127 – 174

Aim

Apply theorems of limits and formulas of derivatives to solve problems including trigonometric functions, optimisation, and motion.

Objectives

After completing this unit, the learners should be able to:

- Find the domain and range of trigonometric function and their inverses.
- Study the parity of trigonometric functions.
- Study the periodicity of trigonometric functions.
- Evaluate limits of trigonometric functions.
- Differentiate trigonometric functions.

Materials

Exercise books, pens, instruments of geometry, calculator

Contents

5.1. Generalities on trigonometric functions and their inverses

Recommended teaching periods: 14 periods

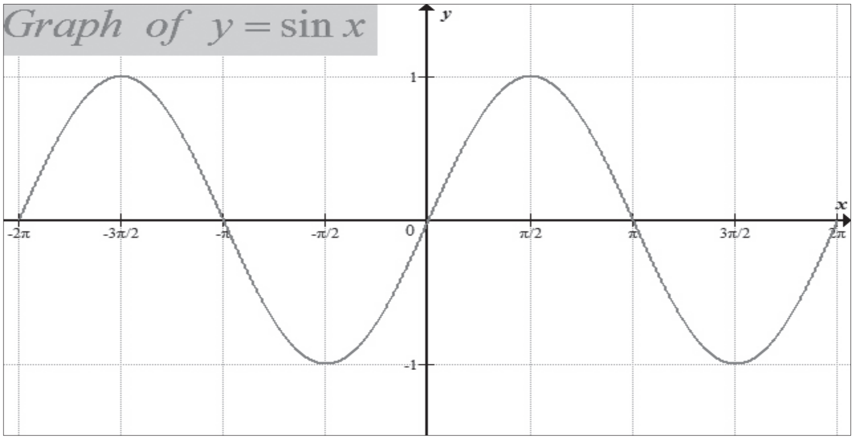
This section looks at the **domain and range of trigonometric functions** and **their inverses**. It also looks at the parity and periodicity of trigonometric functions.

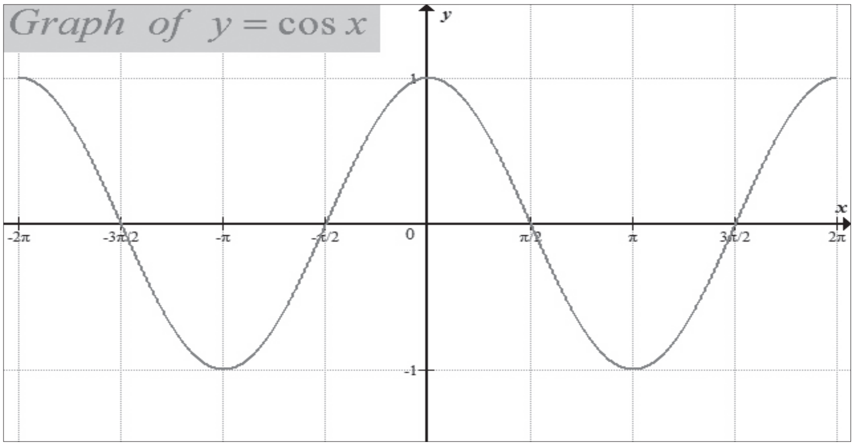
Domain and range of trigonometric functions

Cosine and sine

The domain of $\sin x$ and $\cos x$ is the set of real numbers.

The range of $\sin x$ and $\cos x$ is $[-1,1]$.

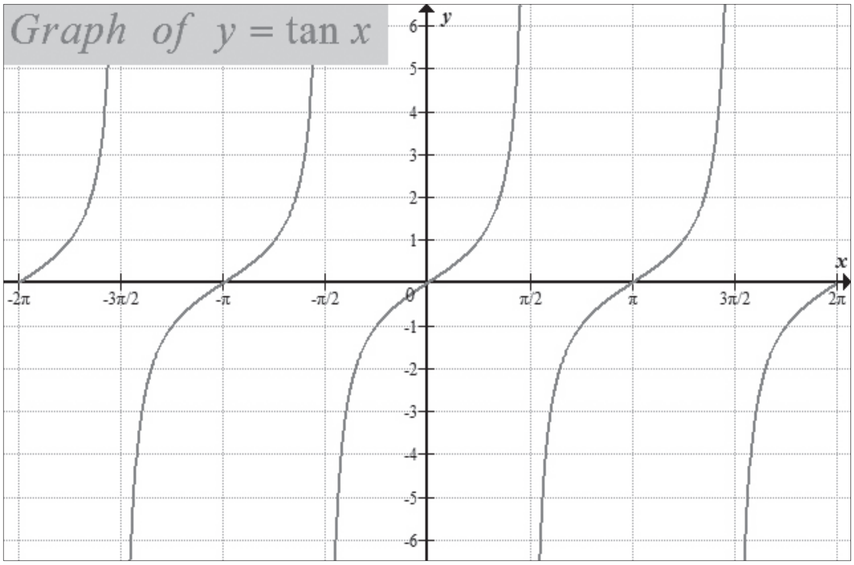


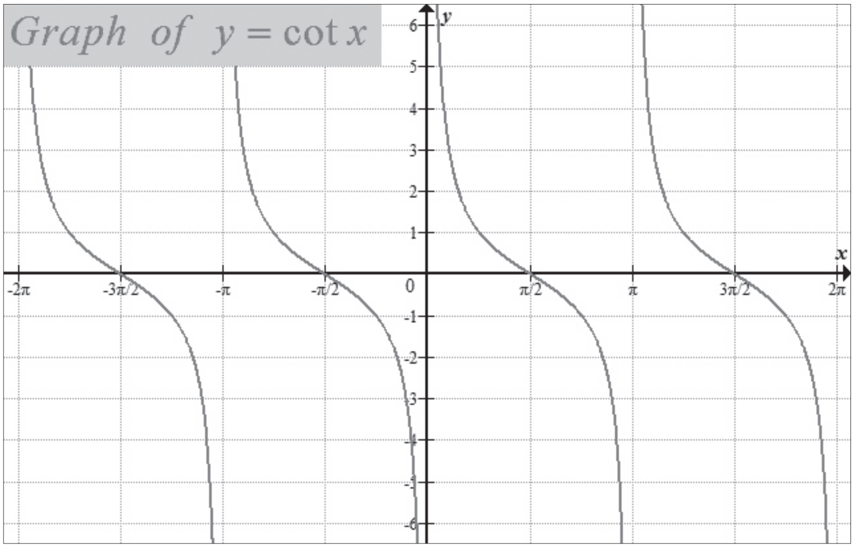


Tangent and cotangent

The domain of $\tan x$ is $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\}, k\in\mathbb{Z}$. The range of $\tan$ is the set of real numbers.

The domain of $\cot x$ is $\mathbb{R}\setminus\{k\pi\}, k\in\mathbb{Z}$. The range of $\cot x$ is the set of real numbers.

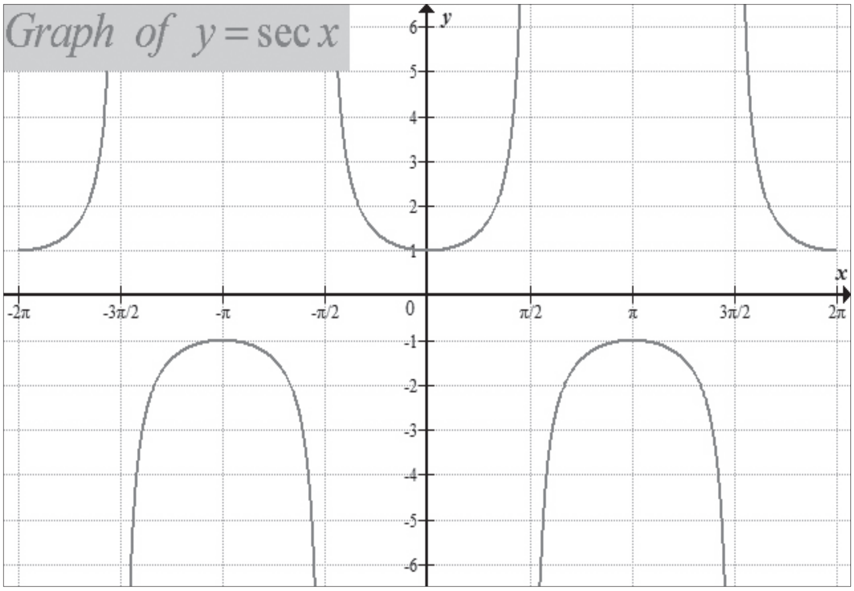


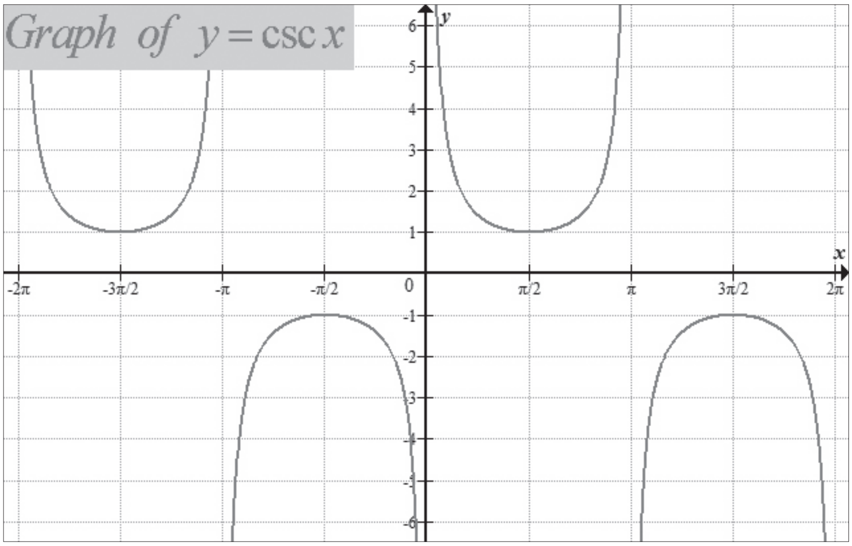


Secant and cosecant

The domain of $\sec x$ is $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}, k \in \mathbb{Z}$. Since $\sec x = \frac{1}{\cos x}$ and range of cosine is $[-1,1]$, $\frac{1}{\cos x}$ will vary from negative infinity to -1 or from 1 to plus infinity. Thus, the range of $\sec x$ is $]-\infty, -1] \cup [1, +\infty[$

The domain of $\csc x$ is $\mathbb{R}\setminus\{k\pi\}, k\in\mathbb{Z}$. Since $\csc x = \frac{1}{\sin x}$ and range of sine is $[-1,1]$, $\frac{1}{\sin x}$ will vary from negative infinity to -1 or from 1 to plus infinity. Thus, the range of $\csc x$ is $]-\infty,-1]\cup[1,+\infty[$



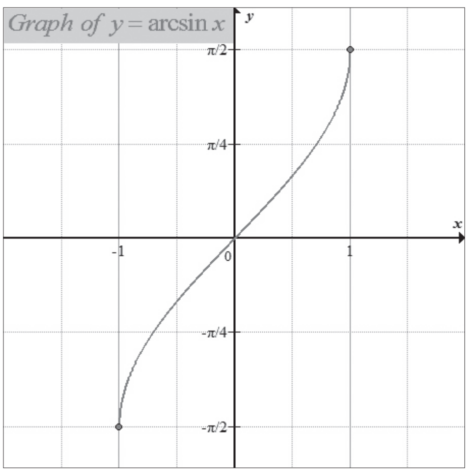


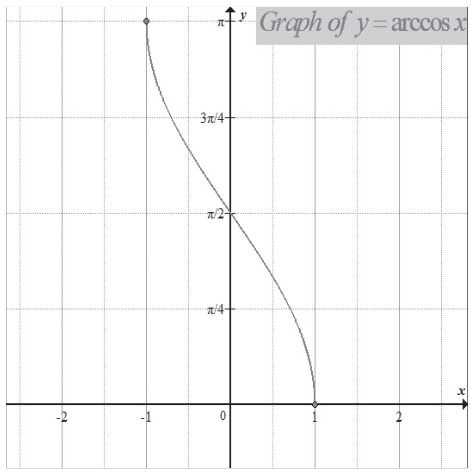
Inverse trigonometric functions

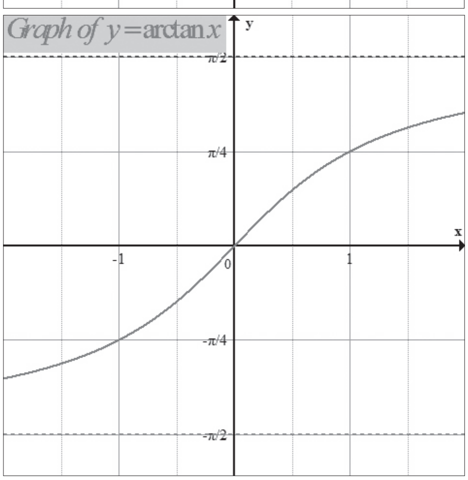
$\sin x$ and $\cos x$ have the inverses called inverse sine and inverse cosine denoted by $\sin^{-1} x$ and $\cos^{-1} x$ respectively.

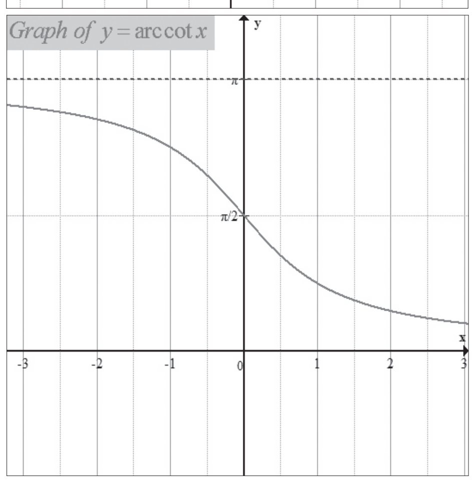
Note that the symbols $\sin^{-1} x$ and $\cos^{-1} x$ are never used to denote $\dfrac{1}{\sin x}$ and $\dfrac{1}{\cos x}$ respectively. If desired, $\dfrac{1}{\sin x}$ and $\dfrac{1}{\cos x}$ can be written as $(\sin x)^{-1}$ and $(\cos x)^{-1}$ (or $\csc x$ and $\sec x$) respectively.

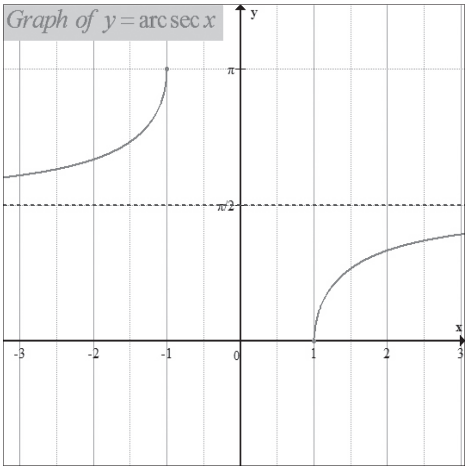
In older literature, $\sin^{-1} x$ and $\cos^{-1} x$ are called arcsine of x and arccosine of x and they are denoted by $arc\sin x$ and $arc\cos x$ respectively.

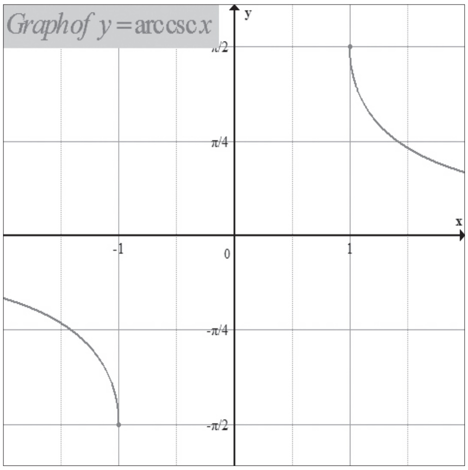












Domain restrictions that make the trigonometric functions one to one

Function	Domain restriction	Range
Sine	$\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$	[-1,1]
Cosine	$]0,\pi[$	[-1,1]
Tangent	$\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$	$\mathbb{R}$
Cotangent	$]0,\pi[$	$\mathbb{R}$
Secant	$\left[0,\frac{\pi}{2}\right[\cup\left]\frac{\pi}{2},\pi\right]$	$]-\infty,-1]\cup[1,+\infty[$
Cosecant	$\left[-\frac{\pi}{2},0\right[\cup\left]0,\frac{\pi}{2}\right]$	$]-\infty,-1]\cup[1,+\infty[$

Because $\sin x$ (restricted) and $\sin^{-1} x$; $\cos x$ (restricted) and $\cos^{-1} x$ are inverses to each other, it follows that:

- $\sin^{-1}(\sin y) = y$ if $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$; $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
- $\cos^{-1}(\cos y) = y$ if $0 \le y \le \pi$; $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$

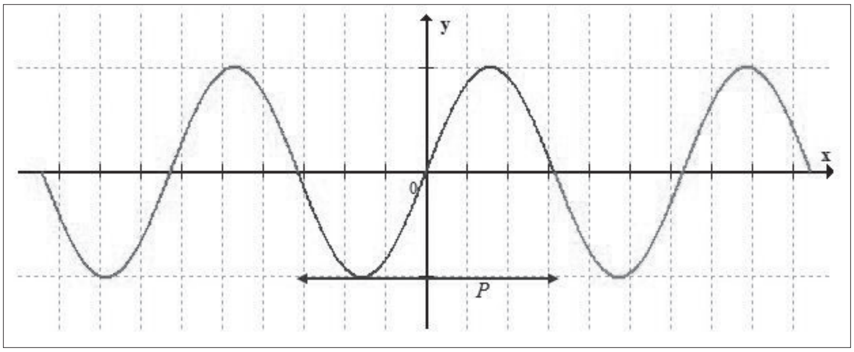
From these relations, we obtain the following important result:

Theorem 1

- If $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then $y = \sin^{-1} x$ and $\sin y = x$ are equivalent.
- If $-1 \le x \le 1$ and $0 \le y \le \pi$, then $y = \cos^{-1} x$ and $\cos y = x$ are equivalent.

Periodic functions

A function f is called periodic if there is a positive number P such that $f(x+P) = f(x)$ whenever x and $x+P$ lie in the domain of f.



Any function which is not periodic is called **aperiodic.**

The period of sum, difference or product of trigonometric function is given by the ***Lowest Common Multiple (LCM)*** of the periods of each term or factor.

Teaching guidelines

Let learners know how to find domain of definition of a polynomial, rational and irrational functions. Recall that the domain of definition of a function is the set of elements where the function is defined.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 5.1 Page 128

Materials

Exercise book, pens, calculator

Answers

1.	No real number for x	2.	No real number for x
3.	$x=\frac{\pi}{2}+k\pi, k\in\mathbb{Z}$	4.	$x=k\pi, k\in\mathbb{Z}$
5.	$x=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$	6.	$x=k\pi,\ k\in\mathbb{Z}$

Exercise 5.1 Page 131

1.	$\mathbb{R}$	2.	$\mathbb{R}\setminus\{0\}$
3.	$\mathbb{R}\setminus\{0\}$	4.	$\mathbb{R}\setminus\{0\}$
5.	$\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\}, k\in\mathbb{Z}$	6.	$]0,+\infty[$

Activity 5.2 Page 132

Materials

Exercise book, pens, calculator

Answers

1.	$]-\infty,-1[\cup]1,+\infty[$	2.	$]-\infty,-1[\cup]1,+\infty[$
3.	No real number for x	4.	No real number for x
5.	$]-1,1[$	6.	$]-1,1[$

Exercise 5.2 Page 139

1.	$\left[-\frac{1}{2},0\right[\cup\left]0,\frac{1}{2}\right]$	2.	$[-1,1]$
3.	$]0,1]$	4.	$[-1,0[\cup]0,1]$

Activity 5.3 Page 139

Materials

Exercise book, pens, calculator

Answers

1. $f(-x)=\frac{\sin x}{x}, -f(x)=-\frac{\sin x}{x}, f(-x)\neq -f(x), f(-x)=f(x)$
2. $g(-x)=-\frac{\cos x}{x}, -g(x)=-\frac{\cos x}{x}, g(-x)=-g(x), g(-x)\neq g(x)$
3. $h(-x)=-\sin x+\cos x, -h(x)=-\sin x-\cos x, h(-x)\neq -h(x), h(-x)\neq h(x)$

Exercise 5.3 Page 141

1. Even 2. Odd 3. Odd
4. Neither even nor odd, 1 is in domain but -1 is not in domain.

Activity 5.4 Page 141

Materials

Exercise book, pens, calculator

Answers

1.	$2k\pi,\ k\in\mathbb{Z}$	2.	$2k\pi,\ k\in\mathbb{Z}$	3.	$k\pi,\ k\in\mathbb{Z}$

Exercise 5.4 Page 144

1. π	2. 3π	3. $\frac{\pi}{3}$
4. 2π	5. $\frac{2\pi}{w}$	6. $\frac{\pi}{2}$

Activity 5.5 Page 144

Materials

Exercise book, pens, calculator

Answers

1. 2π	2. π	3. 2π

Exercise 5.5 Page 147

1. π	2. 2π
3. 2π	4. $\frac{2\sqrt{3}\pi}{3}$

5.2. Limits of trigonometric functions

Recommended teaching periods: 9 periods

This section looks at the method used to find the limits of trigonometric functions

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 5.6 Page 148

Materials

Exercise book, pens, calculator

Answers

1. a) $\lim\limits_{x\to 0}\sin x = \sin 0 = 0$ b) $\lim\limits_{\to} x\sin x = 0\sin 0 = 0$

c) $\lim\limits_{x\to 0}\cos x = \cos 0 = 1$ d) $\lim\limits_{x\to 0}\frac{1}{x} = \infty$

e) $\lim\limits_{x\to 0}\frac{\cos x}{x} = \lim\limits_{x\to 0}\frac{1}{x}\lim\limits_{x\to 0}\cos x = \infty\times 1 = \infty$

2.

x	$\frac{\sin x}{x}$
1	0.841470985
0.9	0.870363233
0.8	0.896695114
0.7	0.920310982
0.6	0.941070789
0.5	0.958851077
0.4	0.973545856
0.3	0.985067356
0.2	0.993346654

x	$\frac{\sin x}{x}$
-1	0.841470985
-0.9	0.870363233
-0.8	0.896695114
-0.7	0.920310982
-0.6	0.941070789
-0.5	0.958851077
-0.4	0.973545856
-0.3	0.985067356
-0.2	0.993346654

0.1	0.998334166
0.01	0.999983333
0.001	0.999999833
0.0001	0.999999998

-0.1	0.998334166
-0.01	0.999983333
-0.001	0.999999833
-0.0001	0.999999998

a) $\lim_{x\to 0^+}\frac{\sin x}{x}=1$ b) $\lim_{x\to 0^-}\frac{\sin x}{x}=1$

c) Since both limits on each side are equal to 1

then $\lim_{x\to 0}\frac{\sin x}{x}=1$

Exercise 5.6 Page 154

1. $\frac{\pi}{4}$ 2. 2 3. 0 4. 3

Activity 5.7 Page 154

Materials

Exercise book, pens, calculator

Answers

1. a) Let $y=\sin^{-1}(-1)$. This equation is equivalent to $\sin y=-1, -\frac{\pi}{2}\le y\le\frac{\pi}{2}$. The only value of y satisfying these conditions is $y=-\frac{\pi}{2}$. So $\sin^{-1}(-1)=-\frac{\pi}{2}$

b) Let $y=\tan^{-1}1$. This is equivalent to

$\tan y=1, -\frac{\pi}{2}<y<\frac{\pi}{2}$. The only value of y satisfying these conditions is $y=\frac{\pi}{4}$. So $\tan^{-1}(1)=\frac{\pi}{4}$

c) Let $y=\cot^{-1}(-1)$. This is equivalent to

$\cot y=-1, 0<y\le\pi$. The only value of y satisfying these conditions is $y=\frac{3\pi}{4}$. So $\cot^{-1}(-1)=\frac{3\pi}{4}$

d) Let $y = \sec^{-1}(-2)$. This is equivalent to

$\sec y = -2,\ 0 \le y \le \pi,\ y \ne \frac{\pi}{2}$. The only value of y satisfying these conditions is $y = \frac{2\pi}{3}$. So $\sec^{-1}(-2) = \frac{2\pi}{3}$

e) Let $y = \csc^{-1}(-2)$. This is equivalent to

$\csc y = -2,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$. The only value of y satisfying these conditions is $y = -\frac{\pi}{6}$. So $\csc^{-1}(-2) = -\frac{\pi}{6}$

2. a) $\lim_{x\to 1} \cot^{-1}\left(\frac{2x-3}{x}\right) = \cot^{-1}\left(\frac{2-3}{1}\right) = \cot^{-1}(-1) = \frac{3\pi}{4}$

b) $\lim_{x\to 1} \sin^{-1}\left(\frac{1+x}{2x}\right) = \sin^{-1}\left(\frac{1+1}{2}\right) = \sin^{-1}(1) = \frac{\pi}{2}$

c) $\lim_{x\to 0} \cos^{-1}\left(\frac{\sqrt{x+1}-1}{x}\right) = \cos^{-1}\left(\frac{0}{0}\right)$ I.C.

Remove this I.C

$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0}\frac{1}{2\sqrt{x+1}} = \frac{1}{2}$ and $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$.

Thus, $\lim_{x\to 0} \cos^{-1}\left(\frac{\sqrt{x+1}-1}{x}\right) = \frac{\pi}{3}$

d) $\lim_{x\to -1} \tan^{-1}\left(\frac{1-x^2}{2x+2}\right) = \tan^{1}\frac{0}{0}$ I.C. Remove this I.C

$\lim_{x\to -1}\left(\frac{1-x^2}{2x+2}\right) = \lim_{x\to -1}\frac{-2x}{2} = 1$ and $\tan^{-1}(1) = \frac{\pi}{4}$.

Thus, $\lim_{x\to -1} \tan^{-1}\left(\frac{1-x^2}{2x+2}\right) = \frac{\pi}{4}$

Exercise 5.7 Page 156

1) $\frac{\pi}{4}$ 2) $-\frac{\pi}{2}$ 3) 1 4) $\frac{1}{2}$ 5) 5 6) 2

5.3. Differentiation of trigonometric functions and their inverses

Recommended teaching periods: 14 periods

This section looks the derivative of trigonometric functions and their inverses.

Derivative of tri atgonometric functions

1. $\frac{d(\sin u)}{dx}=\cos u\frac{du}{dx}$
2. $\frac{d(\cos u)}{dx}=-\sin u\frac{du}{dx}$
3. $\frac{d(\tan u)}{dx}=\sec^2 u\frac{du}{dx}$
4. $\frac{d\left(\cot u\right)}{dx}=-\csc^2 u\frac{du}{dx}$
5. $\frac{d(\sec u)}{dx}=\sec u\tan u\frac{du}{dx}$
6. $\frac{d\left(\csc u\right)}{dx}=-\csc u\cot u\frac{du}{dx}$

Derivative of inverse trigonometric functions

1. $\frac{d\left(\arcsin u\right)}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
2. $\frac{d\left(\arccos u\right)}{dx}=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
3. $\frac{d\left(\arctan u\right)}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$
4. $\frac{d\left(\text{arccot}\, u\right)}{dx}=-\frac{1}{1+u^2}\frac{du}{dx}$
5. $\frac{d\left(\text{arcsec}\, u\right)}{dx}=\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$
6. $\frac{d\left(\text{arccsc}\, u\right)}{dx}=-\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Teaching guidelines

Let learners know trigonometric identities. The trigonometric identities will be used in this section

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 5.8 Page 156

Materials

Exercise book, pens

Answers

1. $\forall x_0 \in \mathbb{R}$

$$(\sin x_0)' = \lim_{x\to x_0} \frac{\sin x - \sin x_0}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{2\cos\frac{x+x_0}{2}\sin\frac{x-x_0}{2}}{x-x_0}$$

$$= \lim_{x\to x_0} \cos\frac{x+x_0}{2} \lim_{x\to x_0} \frac{2\sin\frac{x-x_0}{2}}{x-x_0}$$

$$= \lim_{x\to x_0} \cos\frac{x+x_0}{2} \lim_{x\to x_0} \frac{2\sin\frac{x-x_0}{2}}{2\frac{x-x_0}{2}}$$

$$= \lim_{x\to x_0} \cos\frac{x+x_0}{2} \lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}}{\frac{x-x_0}{2}}$$

$$= \left(\cos\frac{x_0+x_0}{2}\right)\times 1$$

$$= \cos x_0$$

Thus, $\forall x \in \mathbb{R},\ (\sin x)' = \cos x$

2. $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$(\cos x)' = \left[\sin\left(\frac{\pi}{2} - x\right)\right]'$$
$$= \left(\frac{\pi}{2} - x\right)' \cos\left(\frac{\pi}{2} - x\right)$$
$$= -\cos\left(\frac{\pi}{2} - x\right)$$
$$= -\sin x$$

Thus, $\forall x \in \mathbb{R},\ (\cos x)' = -\sin x$

Exercise 5.8 Page 157

1. $2x\cos(x^2 + 3)$
2. $6x\cos\left(x^2 + 4\right)\sin^2\left(x^2 + 4\right)$
3. $-6x\sin 3x^2$
4. $-6\cos^2 2x \sin 2x$

Activity 5.9 Page 157

Materials

Exercise book, pens

Answers

1. $\tan x = \frac{\sin x}{\cos x}$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$
$$= \frac{(\sin x)' \cos x - \sin x(\cos x)'}{(\cos x)^2}$$
$$= \frac{\cos x \cos x + \sin x \sin x}{(\cos x)^2}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$

$$\begin{aligned}(\tan x)' &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\\ &= 1+\tan^2 x\end{aligned}$$

2. $$\cot x = \tan\left(\frac{\pi}{2}-x\right)$$

$$\begin{aligned}(\cot x)' &= \left[\tan\left(\frac{\pi}{2}-x\right)\right]'\\ &= \frac{\left(\frac{\pi}{2}-x\right)'}{\cos^2\left(\frac{\pi}{2}-x\right)}\\ &= \frac{-1}{\sin^2 x}\end{aligned}$$

$$\begin{aligned}(\cot x)' &= \frac{-1}{\sin^2 x}\\ &= -\csc^2 x\\ &= -\left(1+\cot^2 x\right)\end{aligned}$$

Exercise 5.9 Page 159

1. $\tan x + x\left(1+\tan^2 x\right)$
2. $3\left[1+\tan^2(3x+2)\right]$
3. $-2x\left[1+\cot^2\left(x^2-5\right)\right]$
4. $-4\sin x\left(1+\cot^2 4x\right)+\cos x\cot 4x$

Activity 5.10 Page 159

Materials

Exercise book, pens

Answers

1. $$\sec x = \frac{1}{\cos x}$$

$$\begin{aligned}(\sec x)' &= \left(\frac{1}{\cos x}\right)'\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{1}{\cos x}\,\frac{\sin x}{\cos x}\\ &= \sec x\tan x\end{aligned}$$

2. $\csc x = \dfrac{1}{\sin x}$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)'$$
$$= \frac{-\cos x}{\sin^2 x}$$
$$= \frac{-1}{\sin x}\frac{\cos x}{\sin x}$$
$$= -\csc x \cot x$$

Exercise 5.10 Page 160

1. $3\sec(3x+2)\tan(3x+2)$ 2. $\theta^2 \csc 2\theta (3 - 2\theta \cot 2\theta)$
3. $12\sec^4 3x \tan 3x$

Activity 5.11 Page 160

Materials

Exercise book, pens

Answers

1. $f(x) = \sin^{-1} x$ for $x \in [-1,1]$ and $x = \sin y$ for $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ where $y = f(x)$.

$$(\sin^{-1} x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$
$$= \frac{1}{\sqrt{1-\sin^2(\sin^{-1} x)}} \quad \text{since } \cos x = \sqrt{1-\sin^2 x}$$
$$= \frac{1}{\sqrt{1-x^2}}$$

2. $f(x) = \cos^{-1} x$ for $x \in [-1,1]$ and $x = \cos y$ for $y \in [0, \pi]$ where $y = f(x)$

$$\left(\cos^{-1} x\right)' = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = \frac{-1}{\sin\left(\cos^{-1} x\right)}$$

$$= \frac{-1}{\sqrt{1-\cos^2\left(\cos^{-1} x\right)}} \quad \text{since } \sin x = \sqrt{1-\cos^2 x}$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

Exercise 5.11 Page 162

1. $\dfrac{1}{|x|\sqrt{x^2-1}}$
2. $\dfrac{-2x}{\sqrt{1-x^4}}$
3. $\dfrac{-1}{\sqrt{2x-x^2}}$
4. $\dfrac{1}{\sqrt{2x}\sqrt{1-2x}}$

Activity 5.12 Page 162

Materials

Exercise book, pens

Answers

1. $f(x) = \tan^{-1} x$ for $x \in \mathbb{R}$ and $x = \tan y$ for $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ where $y = f(x)$.

$$\left(\tan^{-1} x\right)' = \frac{1}{(\tan y)'} = \frac{1}{1+\tan^2 y} = \frac{1}{1+\tan^2\left(\tan^{-1} x\right)} = \frac{1}{1+x^2}$$

2. $f(x) = \cot^{-1} x$ for $x \in \mathbb{R}$ and $x = \cot y$ for $y \in]0, \pi[$ where $y = f(x)$

$$\left(\cot^{-1} x\right)' = \frac{1}{(\cot y)'} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+\cot^2\left(\cot^{-1} x\right)} = \frac{-1}{1+x^2}$$

Exercise 5.12 Page 163

1. $\dfrac{1}{2\sqrt{x}(1+x)}$
2. $\dfrac{1}{|x|\sqrt{x^2-1}} + \dfrac{1}{x^2+1}$
3. $\dfrac{-1}{2x\sqrt{x-1}}$

Activity 5.13 Page 163

Materials

Exercise book, pens

Answers

1. $f(x)=\sec^{-1}x$ for $x\le -1$ or $x\ge 1$ and $x=\sec y$ for $y\in[0,\pi], y\ne \frac{\pi}{2}$ where $y=f(x)$

$$\left(\sec^{-1}x\right)'=\frac{1}{(\sec y)'}=\frac{1}{\sec y\tan y}=\frac{1}{x\sqrt{x^2-1}}$$

2. $f(x)=\csc^{-1}x$ for $x\le -1$ or $x\ge 1$ and $x=\csc y$ for $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right], y\ne 0$ where $y=f(x)$

$$\left(\csc^{-1}x\right)'=\frac{1}{(\csc y)'}=\frac{-1}{\csc y\cot y}=\frac{-1}{x\sqrt{x^2-1}}$$

Exercise 5.13 Page 164

1. $\dfrac{1}{(2x+1)\sqrt{x^2+x}}$
2. $\dfrac{1}{x\sqrt{x^2-1}}-\dfrac{1}{|x|\sqrt{x^2-1}}$
3. $\dfrac{1}{x\sqrt{25x^2-1}}$
4. $\dfrac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$

Activity 5.14 Page 165

Materials

Exercise book, pens

Answers

1. $-2\sin 2x$
2. $-4\cos 2x$
3. $8\sin 2x$
4. $16\cos 2x$
5. $-32\sin 2x$

Exercise 5.14 Page 166

1. a) $-13\sin 2x\sin 3x+12\cos 2x\cos 3x$

   b) $-\dfrac{3x+1}{4x\left(1+x^2\right)\sqrt{x}}$ c) $2\cos 2x$

   d) $\dfrac{2}{\cos^4 x}\left(1+2\sin^2 x\right)$

2. a) $\dfrac{-8x}{\left(1+x^2\right)^3}$ b) $-62\cos 2x\cos 3x+63\sin 2x\sin 3x$

   c) $\dfrac{2\sin x}{\left(4\cos^2 x-1\right)^4}\left(-64\cos^6 x+16\cos^4 x+484\cos^2 x+23\right)$

   d) $\dfrac{16\tan x}{\left(\tan^2 x-1\right)^4}\left(5\tan^6 x+19\tan^4 x+19\tan^2 x+5\right)$

3. a) $\sin\left(x+\dfrac{n\pi}{2}\right)+\cos\left(x+\dfrac{n\pi}{2}\right)$ b) $2^n\cos\left(2x+\dfrac{n\pi}{2}\right)$

5.4. Applications

Activity 5.15 Page 167

Materials

Exercise book, pens

Answers

By reading text books or accessing internet, learners will discuss on harmonic motion and how differentiation of trigonometry functions is used to find velocity, acceleration and jerk of an object if the function representing its position is known.

If we have the function representing the position, say $S(t)$, then

- The velocity of the object is $v=\dfrac{ds}{dt}$

- The acceleration of the object is $a = \frac{d^2 s}{dt^2}$
- The jerk of the object is $j = \frac{d^3 s}{dt^3}$

Exercise 5.15 Page 168

1. a) $3m$ b) $-9\pi\, {}^{m}/_{s}$ c) $-27\pi\, {}^{m}/_{s^2}$

   d) $\left(3\pi t + \frac{\pi}{3}\right)$ e) $\frac{3}{2}Hz$ f) $\frac{2}{3}$

2. a) amplitude x_m is $4m$

   frequency f is $0.5\,Hz$

   period T is 2

   angular frequency ω is π

   b) velocity is $\frac{dx}{dt} = -4\pi \sin\left(\pi t + \frac{\pi}{4}\right)$

   acceleration $\frac{d^2x}{dt^2} = -4\pi^2 \cos\left(\pi t + \frac{\pi}{4}\right)$

   c) displacement at $t = 1$ is $-2\sqrt{2}\pi\, m$

   velocity at $t = 1$ is $2\pi\sqrt{2}\ {}^{m}/_{s}$

   acceleration at $t = 1$ is $2\pi^2\sqrt{2}\ {}^{m}/_{s^2}$

   d) the maximum speed $4\pi\, {}^{m}/_{s}$

   maximum acceleration $4\pi^2\, {}^{m}/_{s^2}$

End of Unit Assessment Page 171

1. a) 2π b) π c) 2π
   d) not periodic (aperiodic)
   e) π f) not period (aperiodic)
2. a) neither even nor odd b) even c) odd
   d) odd
3. a) 2 b) $\frac{3}{2}$ c) 1 d) 2
   e) 7 f) $\frac{2}{3}$ g) $\frac{1}{16}$ h) $\frac{-11}{9}$
   i) $\frac{15}{7}$ j) $\frac{1}{2}$ k) 1 l) $\cos a$
   m) $-\sin a$ n) 0 o) 4 p) 0
   q) 2 r) $\frac{1}{2}$ s) $\frac{3}{4}$ t) $\frac{1}{2}$
   u) $\frac{\pi^2}{2}$ v) 0 w) $\frac{\sqrt{2}}{32}$
4. a) $3\sec x\tan x+10\csc^2 x$ b) $-12x^{-5}-2x\tan x-x^2\sec^2 x$
   c) $5\cos 2x-4\csc x\cot x$ d) $\frac{3\cos t-2}{\left(3-2\cos t\right)^2}$
   e) $-48x\sin\left(6x^2+5\right)$ f) $72x^3\sin^2\left(2x^4+1\right)\cos\left(2x^4+1\right)$
   g) $4\left(x-\cos^2 x\right)^3\left(1+\sin 2x\right)$ h) $\frac{2\sin 4x-4\left(2x+3\right)\cos 4x}{\sin^2 4x}$
   i) $\frac{-2x^2}{\sqrt{1-x^2}}$ j) $\frac{-1}{\sqrt{1-x^2}}$
   k) $\frac{-2}{x\sqrt{x^2-4}}$ l) $\frac{x^2-1}{x\sqrt{x^2-1}}$
   m) $\sin^{-1} x$

5. The amount of money in the bank account will be increasing during the following intervals:
   $2.1588 < t < 5.3004$, $8.4420 < t < 10$
6. $-\sqrt{2}m/\sec,\ \sqrt{2}m/\sec,\ \sqrt{2}m/\sec^2,\ \sqrt{2}m/\sec^2$
7. $0cm/\sec,\ -5\sqrt{3}cm/\sec,\ -5\sqrt{2}cm/\sec$
8. $4m/\sec,\ -4m/\sec,\ -4m/\sec$
9. It multiplies the velocity, acceleration, and jerk by 2, 4, and 8, respectively
10. a) $\frac{\pi}{3}$ b) $-\frac{\pi}{4}$ c) $-\frac{\pi}{3}$ d) $\frac{3\pi}{4}$
    e) $\frac{6}{7}$ f) 1 g) 1 h) 1

Unit 6 Vector Space of Real Numbers

Learner's Book page 175 – 222

Aim

Apply vectors of $\mathbb{R}^3$ to solve problems related to angles using the scalar product in $\mathbb{R}^3$ and use the vector product to solve also problems in $\mathbb{R}^3$.

Objectives

After completing this unit, the learners should be able to:

- define and apply different operations on vectors.
- show that a vector is a sub-vector space.
- define linear combination of vectors.
- find the norm of a vector.
- calculate the scalar product of two vectors.
- calculate the angle between two vectors.
- apply and transfer the skills of vectors to other area of knowledge.

Materials

Exercise books, pens, instruments of geometry, calculator

Contents

6.1. Vector space $\mathbb{R}^3$

Recommended teaching periods: 9 periods

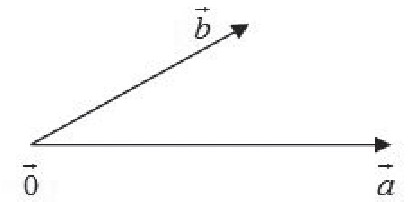
This section looks at the **operations on vectors in space**, **sub-vector spaces** and **linear combination of vectors**

The set of vectors of space with origin 0 is denoted by E_0 and $E_0 = \left\{\overrightarrow{0a} : a \in E\right\}$.

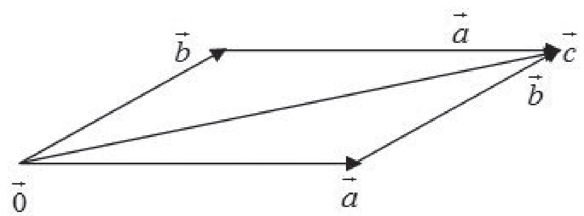
6.2. Operations on vectors

Sum of two vectors

Two non-parallel (or opposite) vectors of the same origin (means that their tails are together) determine one and only one plane in space.



The addition of vectors of E_0 is the application defined by $E_0 \times E_0 \to E_0$



$\vec{c}$ is then the diagonal of the parallelogram built from $\vec{a}$ and $\vec{b}$. Thus, $\vec{a}+\vec{b}=\vec{c}$

If $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$, $\vec{a}+\vec{b}=(a_1+b_1,a_2+b_2,a_3+b_3)$

A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by rational numbers, or generally any field.

If $(\mathbb{R},F,+)$ is a subspace of $(\mathbb{R},E,+)$, then

- $F \subset E$
- $0-vector \in F$
- $\vec{u},\vec{v} \in F, \alpha, \beta \in \mathbb{R}; \quad \alpha\vec{u}+\beta\vec{v} \in F$

Sum of two sub-vector spaces

If F and G are two sub-vector spaces of E then the sum of F and G is also a sub-vector space of E. It is denoted as $F+G=\{x+y, x\in F, y\in G\}$

Theorems

- W_1 and W_2 are subspaces of V, then $W_1 \cup W_2$ is a subspace $\Leftrightarrow W_1 \subseteq W_2 \text{ or } W_2 \subseteq W_1$.
- W_1 and W_2 are subspace of V, then $W_1 + W_2$ is the smallest subspace that contains both W_1 and W_2.

Property

If $(\mathbb{R}, F, +)$ and $(\mathbb{R}, G, +)$ are two sub-vector spaces of $(\mathbb{R}, E, +)$ we have, $\dim(F+G)=\dim(F)+\dim(G)-\dim(F\cap G)$.

Remark

If $\dim(F\cap G)=0$, then $\dim(F+G)=\dim(F)+\dim(G)$. In this case, F and G are said to be **complementary** and the sum $F+G$ is said to be a **direct sum**; and it is denoted by $F\oplus G$. Otherwise, F and G are said to be **supplementary**.

6.3. Linear combination

- The vector u is called a linear combination of the vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$ provided that there exists scalars c_1, c_2, c_3 such that $\vec{u}=c_1\vec{u_1}+c_2\vec{u_2}+c_3\vec{u_3}$
- Let $S=\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ be a set of vectors in the vector space V. The set of all linear combinations of $\vec{u_1}, \vec{u_2}, \vec{u_3}$ is called the **span** of the set S, denoted by $span(S)$ or $span(\vec{u_1}, \vec{u_2}, \vec{u_3})$.
- The set of vectors $S=\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ of a vector space V is said to be **linearly independent** provided that the equation $c_1\vec{u_1}+c_2\vec{u_2}+c_3\vec{u_3}=0$ has only the trivial solution $c_1=c_2=c_3=0$.
- A set of vectors is called **linearly dependent** if it is not linearly independent. Or if $c_1u_1+c_2u_2+c_3u_3=0$ for $c_1, c_2, c_3 \neq 0$.

Theorem 1

The three vectors $\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3}$ in $\mathbb{R}^3$ are linearly independent if and only if the 3×3 matrix $A=\begin{bmatrix}\overrightarrow{u_1} & \overrightarrow{u_2} & \overrightarrow{u_3}\end{bmatrix}$ with the vectors as columns has non zero determinant otherwise they are linearly dependent.

Teaching guidelines

Let learners know how to perform operations on vectors in 2-dimension. In three dimensions, there is a third component, z

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

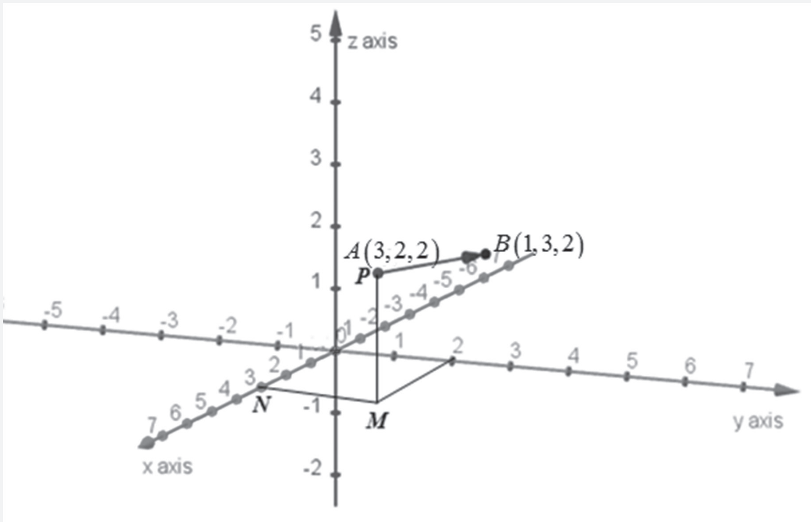
Answers to activities and exercises

Activity 6.1 Page 176

Materials

Exercise book, pens, instruments of geometry

Answers



$B-A=(-2,1,0)$

Exercise 6.1 Page 182

1. $(-3,-3,3)$
2. $(-1,0,8)$
3. $(-14,-17,-11)$
4. $(17,21,23)$

Activity 6.2 Page 183

Materials

Exercise book, pens, calculator

Answers

1. $x=0$
2. $\alpha\vec{u}+\beta\vec{v}=\alpha(0,a,3a)+\beta(0,b,3b)$

$=(0,\alpha a+\beta b,3\alpha a+3\beta b)$

$=(0,\alpha a+\beta b,3(\alpha a+\beta b))$

$=(0,c,3c)\in V,\quad for\quad \alpha a+\beta b=c$

Thus, $\alpha\vec{u}+\beta\vec{v}\in V$

Exercise 6.2 Page 187

1. $F \in \mathbb{R}^3$

   If we take $x=0, y=0$, we see that $(0,0,0) \in F$

   Consider $\vec{k}=(a,b,0)$, $\vec{t}=(c,d,0) \in F$, $\alpha, \beta \in \mathbb{R}$

   $$\alpha\vec{k}+\beta\vec{t}=\alpha(a,b,0)+\beta(c,d,0)$$
   $$=(\alpha a,\alpha b,0)+(\beta c,\beta d,0)$$
   $$=(\alpha a+\alpha c,\beta b+\beta d,0)$$

   F is a subspace of $\mathbb{R}^3$

2. $G \in \mathbb{R}^3$

   If we take $x=0, y=0$, we see that $(0,0,0) \in G$

   Consider $\vec{k}=(a,b,0)$, $\vec{t}=(c,d,0) \in G$, $\alpha, \beta \in \mathbb{R}$

   $$\alpha\vec{k}+\beta\vec{t}=\alpha(2a,2b,0)+\beta(2c,2d,0)$$
   $$=(2\alpha a,2\alpha b,0)+(2\beta c,2\beta d,0)$$
   $$=(2(\alpha a+\alpha c),2(\beta b+\beta d),0)$$

   G is a subspace of $\mathbb{R}^3$

3. $H \in \mathbb{R}^3$

   If we take $x=0, y=0$, we see that $(0,0,0) \in H$

   Consider $\vec{k}=(a,0,b)$, $\vec{t}=(c,0,d) \in H$, $\alpha, \beta \in \mathbb{R}$

   $$\alpha\vec{k}+\beta\vec{t}=\alpha(a,0,b)+\beta(c,0,d)$$
   $$=(\alpha a,0,\alpha b)+(\beta c,0,\beta d)$$
   $$=(\alpha a+\alpha c,0,\beta b+\beta d)$$

   H is a subspace of $\mathbb{R}^3$

4. $K \in \mathbb{R}^3$

   I f we take $x=0, z=0$, we see that $(0,0,0) \notin K$

   Therefore, K is not a subspace of $\mathbb{R}^3$

Activity 6.3 Page 187

Materials

Exercise book, pens, calculator

Answers

$a = 3,\ b = 2$

Exercise 6.3 Page 191

1. $\vec{v} = -6\vec{e_1} + 3\vec{e_2} + 2\vec{e_3}$
2. Set $(a,b,c) = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$

   $\begin{cases} \alpha = a \\ \beta = b - 2a \\ \gamma = c - 2b + a \end{cases}$. Thus, $\vec{u}, \vec{v}$ and $\vec{w}$ generate $\mathbb{R}^3$
3. a) $\{\vec{u}, \vec{v}, \vec{w}\}$ is not a basis of $\mathbb{R}^3$ b) $k = -8$

Activity 6.4 Page 192

Materials

Exercise book, pens, calculator

Answers

$$\begin{cases} a = \frac{13}{8} \\ b = \frac{9}{4} \\ c = -\frac{11}{8} \end{cases}$$

Exercise 6.4 Page 193

1. $[\vec{u}]_V = (3,-2,-5)$, $[\vec{v}]_V = (3,-5,3)$
2. a) $(2,-5,7)$ b) $(c, b-c, a-b)$
3. $(3,-1,2)$
4. Suppose $\vec{v} = r\vec{e_1} + s\vec{e_2} + t\vec{e_3}$; then $[\vec{v}]_e = (r,s,t)$

From the values of $\overrightarrow{e_1}, \overrightarrow{e_2}$ and $\overrightarrow{e_3}$, we have

$$\vec{v} = r(a_1 f_1 + a_2 f_2 + a_3 f_3) + s(b_1 f_1 + b_2 f_2 + b_3 f_3) + t(c_1 f_1 + c_2 f_2 + c_3 f_3)$$

$$\vec{v} = (ra_1 f_1 + sb_1 f_1 + tc_1 f_1) + (ra_2 f_2 + sb_2 f_2 + tc_2 f_2) + (ra_3 f_3 + sb_3 f_3 + tc_3 f_3)$$

$$\vec{v} = f_1(ra_1 + sb_1 + tc_1) + f_2(ra_2 + sb_2 + tc_2) + f_3(ra_3 + sb_3 + tc_3)$$

Hence,

$$\left[\vec{v}\right]_f = (ra_1 + sb_1 + tc_1, ra_2 + sb_2 + tc_2, ra_3 + sb_3 + tc_3)$$

On the other hand

$$\begin{aligned}\left[\vec{v}\right]_e A &= (r,s,t)\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\ &= (ra_1 + sb_1 + tc_1, ra_2 + sb_2 + tc_2, ra_3 + sb_3 + tc_3) \\ &= \left[\vec{v}\right]_f \qquad \textit{as required}\end{aligned}$$

6.4. Euclidian vector space $\mathbb{R}^3$

Recommended teaching periods: 8 periods

This section talks about:

Scalar product of two vectors

The scalar product of two vectors of space is the application $E_0 \times E_0 \to \mathbb{R}$.

Algebraically, the scalar product of vectors $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1,\ b_2,\ b_3)$ of space is defined by $\vec{u} \cdot \vec{v} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Magnitude of a vector

The magnitude of the vector $\vec{u}$ denoted by $\left\|\vec{u}\right\|$ is defined as its length.

If $\vec{u} = (a,b,c)$ then $\left\|\vec{u}\right\| = \sqrt{a^2 + b^2 + c^2}$.

Distance between two points

The distance between two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ denoted, $d(A, B)$ is

$$d(A,B) = \left\|\overrightarrow{AB}\right\| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

The angle between two vectors

Angle between two vectors $\vec{u}$ and $\vec{v}$ is such that

$$\cos\left(\vec{u}, \vec{v}\right) = \frac{\vec{u} \cdot \vec{v}}{\left\|\vec{u}\right\| \cdot \left\|\vec{v}\right\|}$$

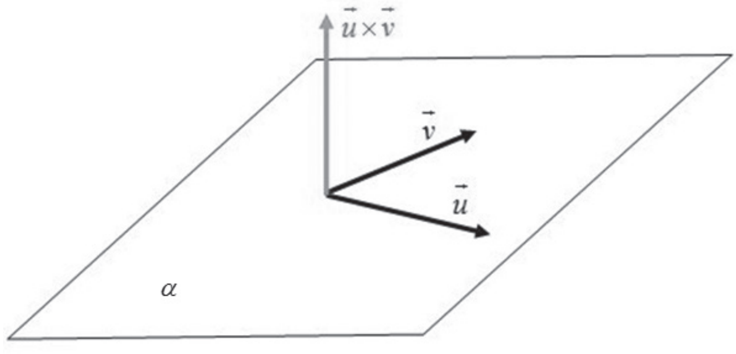
Vector product of two vectors and the mixed product of three vectors.

The vector product (or cross product or Gibbs vector product) of $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1, b_2, b_3)$ is denoted and defined by

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Or

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$



The mixed product (also called the **scalar triple product** or **box product** or **compound product**) of three vectors is a scalar which numerically equals the vector product multiplied by a vector as the dot product.

The vector product of any two vectors is perpendicular to each of these vectors.

Then the mixed product of the vectors $\vec{u}=(a_1,a_2,a_3)$, $\vec{v}=(b_1,b_2,b_3)$ and $\vec{w}=(c_1,c_2,c_3)$ is denoted and defined by

$$\vec{u}\cdot(\vec{v}\times\vec{w})=a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}.$$

Teaching guidelines

Let learners know how to find scalar product of two vectors, magnitude of a vector, angle between two vectors. In three dimensions, there is a third component, z

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 6.5 Page 194

Materials

Exercise book, pens, calculator

Answers

a) 0 b) -3

Exercise 6.5 Page 195

a) 15	b) 18	c) -4	d) 3

Activity 6.6 Page 196

Materials

Exercise book, pens, calculator

Answers

a) $\sqrt{29}$ b) $\sqrt{6}$

Exercise 6.6 Page 198

a) $\sqrt{461}$ unit of length b) $3\sqrt{3}$ unit of length

c) 22 unit of length d) $\sqrt{201}$ unit of length

Activity 6.7 Page 198

Materials

Exercise book, pens, calculator

Answers

1. $\vec{u}\cdot\vec{v}=18$
2. $\|\vec{u}\|\|\vec{v}\|=18$
3. $\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)=2k\pi,\ k\in\mathbb{Z}$

Exercise 6.7 Page 202

1. a) 77.99^0 b) 85.96^0 c) 79.26^0 d) 54.74^0
2. a) $\cos\alpha=\frac{2}{\sqrt{29}},\ \cos\beta=\frac{3}{\sqrt{29}},\ \cos\gamma=\frac{4}{\sqrt{29}}$

   b) $\cos\alpha=\frac{4}{\sqrt{17}},\ \cos\beta=\frac{-1}{\sqrt{17}},\ \cos\gamma=\frac{4}{3\sqrt{17}}$

   c) $\cos\alpha=\frac{1}{\sqrt{201}},\ \cos\beta=\frac{-2}{\sqrt{201}},\ \cos\gamma=\frac{-14}{\sqrt{201}}$

   d) $\cos\alpha=1,\ \cos\beta=0,\ \cos\gamma=0$

Activity 6.8 Page 202

Materials

Exercise book, pens, calculator

Answers

1. Let (a,b,c) be that vector. Using dot product properties

$$\begin{cases}(a,b,c)\cdot(2,-1,3)=0\\(a,b,c)\cdot(1,2,-1)=0\end{cases}\Leftrightarrow\begin{cases}2a-b+3c=0\\a+2b-c=0\end{cases}$$

$$\Leftrightarrow\begin{cases}2a-b+3c=0\\-2a-4b+2c=0\end{cases}$$

$$-5b+5c=0\Rightarrow c=b$$

$$2a-c+3c=0$$

$$\Leftrightarrow 2a+2c=0\Rightarrow a=-c$$

Then $a=-c,\ b=c$

Take $c=1$, we have $\vec{w}=(-1,1,1)$

2. $\vec{i}\begin{vmatrix}-1 & 3\\2 & -1\end{vmatrix}-\vec{j}\begin{vmatrix}2 & 3\\1 & -1\end{vmatrix}+\vec{k}\begin{vmatrix}2 & -1\\1 & 2\end{vmatrix}=-5\vec{i}+5\vec{j}+5\vec{k}$

3. Vector obtained in b is a multiple of vector obtained in a. Or we can say that the two vectors are parallel.

Exercise 6.8 Page 204

1. $(1,-2,5)$	2. $(-2,10,-12)$	3. $(3,3,-3)$
4. $(-66,18,83)$	5. $(8,-18,-14)$	

Activity 6.9 Page 204

Materials

Exercise book, pens, calculator

Answers

1. $\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$ or

$\left(b_2c_3 - c_2b_3, -b_1c_3 + c_1b_3, b_1c_2 - c_1b_2\right)$

2. $\left(a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$ or

$\left(a_1b_2c_3 - a_1c_2b_3, -a_2b_1c_3 + a_2c_1b_3, a_3b_1c_2 - a_3c_1b_2\right)$

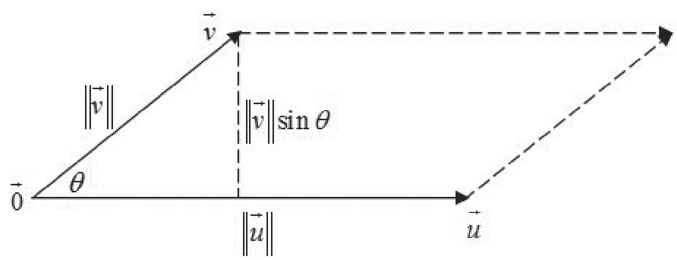
Exercise 6.9 Page 206

1) -4	2) 42	3) 24	4) 320	5) 40

6.5. Applications

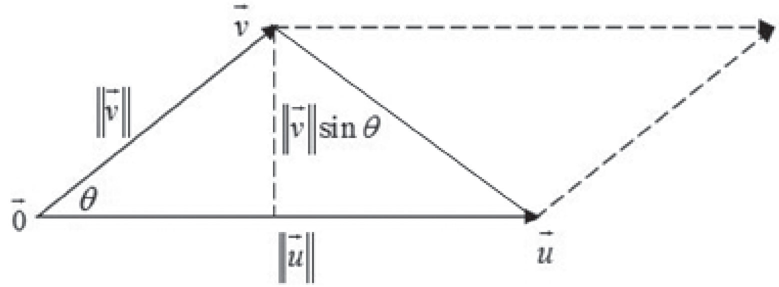
Area of a parallelogram

Area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides is $S_{\square} = \|\vec{u} \times \vec{v}\|$

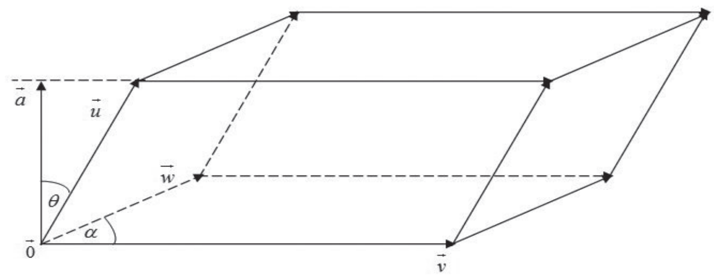


Area of a triangle

Thus, the area of triangle with vectors $\vec{u}$ and $\vec{v}$ as two sides is $S_{\triangle} = \frac{1}{2}\|\vec{u} \times \vec{v}\|$



Volume of a parallelepiped



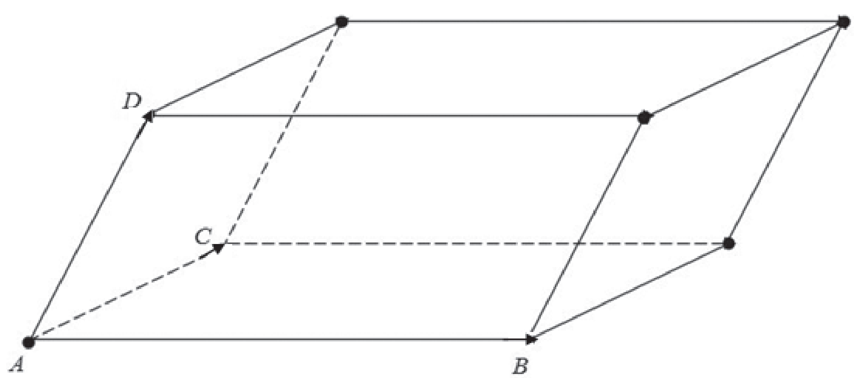
The volume of a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

$V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

Remember that if $\vec{u}=\left(a_1,a_2,a_3\right)$, $\vec{v}=\left(b_1,b_2,b_3\right)$ and $\vec{w}=\left(c_1,c_2,c_3\right)$, then $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

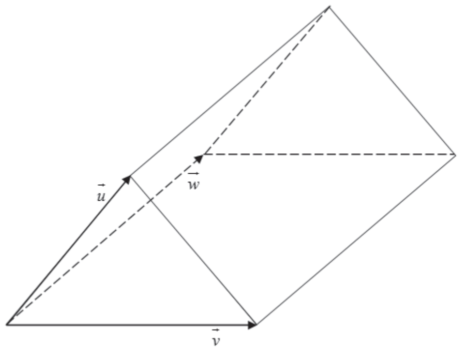
If the parallelepiped is defined by four points $A\left(a_1,a_2,a_3\right)$, $B\left(b_1,b_2,b_3\right)$, $C\left(c_1,c_2,c_3\right)$ and $D\left(d_1,d_2,d_3\right)$, its volume is

$V = \left|\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|$



Volume of a triangular prism

The parallelepiped can be split into 2 triangular prism of equal volume. Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a triangular prism is equal to $\frac{1}{2}$ of the magnitude of the mixed product.

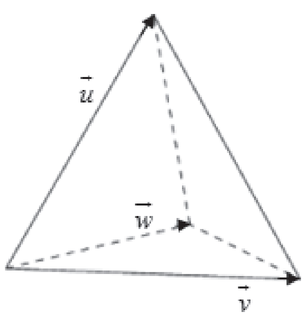


$$V = \frac{1}{2}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Volume of a tetrahedron

The parallelepiped can be split into 6 tetrahedra of equal volume.

Since the volume of a parallelepiped is the magnitude of the mixed product, then the volume of a tetrahedron is equal to $\frac{1}{6}$ of the magnitude of the mixed product.



$$V = \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Remark

A tetrahedron is also called **triangular pyramid**.

Teaching guidelines

Let learners know how to find scalar product, vector product, mixed product and magnitude of a vector.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 6.10 Page 206

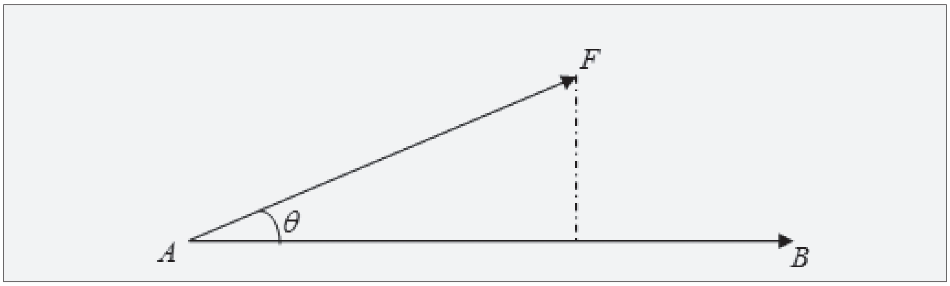
Materials

Exercise book, pens

Answers

If a constant force F acting on a particle displaces it from A to B, then,

$$\begin{aligned} work\ done &= (component\ of\ F)\cdot Displacement \\ &= (F\cos\theta)\cdot AB \\ &= \vec{F}\cdot\overrightarrow{AB} \end{aligned}$$



Exercise 6.10 Page 208

1) 16	2) 55	3) $\frac{141}{2}$
4) 20	5) 32	6) 40

Activity 6.11 Page 208

Materials

Exercise book, pens

Answers

Since the base and the height of this parallelogram are $\left\|\vec{u}\right\|$ and $\left\|\vec{v}\right\|\sin\theta$ respectively, the area is $S_{\square} = \left\|\vec{u}\right\|\left\|\vec{v}\right\|\sin\theta$. But $\left\|\vec{u}\right\|\left\|\vec{v}\right\|\sin\theta = \left\|\vec{u}\times\vec{v}\right\|$. Then area is $S_{\square} = \left\|\vec{u}\times\vec{v}\right\|$.

Thus, the magnitude of the vector product of two vectors $\vec{u}$ and $\vec{v}$ represents the area of a parallelogram with vectors $\vec{u}$ and $\vec{v}$ as two adjacent sides.

Exercise 6.11 Page 210

1. a) $220\sqrt{2}$ sq. units b) $\sqrt{445}$ sq. units
2. $\frac{3\sqrt{2}}{2}$ sq. units
3. $\sqrt{563}$ sq. units
4. $6\sqrt{5}$ sq. units
5. $\sqrt{29}$ sq. units
6. $\frac{\sqrt{6}}{2}$ sq. units

Activity 6.12 Page 211

Materials

Exercise book, pens

Answers

The base of this parallelepiped is defined by the vectors $\vec{v}$ and $\vec{w}$. Then, the area of the base is $S = \left\|\vec{v} \times \vec{w}\right\| = \left\|\vec{v}\right\|\left\|\vec{w}\right\|\sin\alpha$. The height of this parallelepiped is $\left\|\vec{a}\right\|$.

Since the vector $\vec{a}$ is not known we can find the height in terms of $\left\|\vec{u}\right\|$. We see that $\cos\theta = \frac{\left\|\vec{a}\right\|}{\left\|\vec{u}\right\|} \Leftrightarrow \left\|\vec{a}\right\| = \left\|\vec{u}\right\|\cos\theta$.

The angle θ is the angle between the vector $\vec{a}$ and vector $\vec{u}$ but it is also the angle between the vector $\vec{u}$ and the vector given by the vector product $\vec{v}\times\vec{w}$ since this cross product is perpendicular to both $\vec{v}$ and $\vec{w}$.

Now, the volume of the parallelepiped is product of the area of the base and the height.

Then,

$V = \left\|\vec{v}\right\|\left\|\vec{w}\right\|\sin\alpha\left\|\vec{u}\right\|\cos\theta \quad \Leftrightarrow V = \left\|\vec{v}\right\|\left\|\vec{w}\right\|\left\|\vec{u}\right\|\sin\alpha\cos\theta$

$\Leftrightarrow V = \left\|\vec{u}\right\|\left(\left\|\vec{v}\right\|\left\|\vec{w}\right\|\sin\alpha\right)\cos\theta \quad \Leftrightarrow V = \left\|\vec{u}\right\|\left\|\vec{v}\times\vec{w}\right\|\cos\theta$

$\Leftrightarrow V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

Thus, the volume of a parallelepiped which has vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as three concurrent edges, where $\vec{v}$ and $\vec{w}$ define its base, is given by

$V = \left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

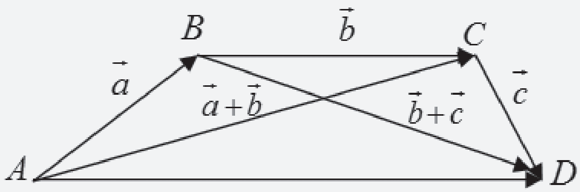
Exercise 6.12 Page 216

1. a) $\frac{3}{2}$ cubic units   b) $\frac{35}{2}$ cubic units
2. a) 2 cubic units   b) $\frac{10}{3}$ cubic units
3. 20 cubic units
4. $7\frac{1}{3}$ cubic units

End of Unit Assessment Page 219

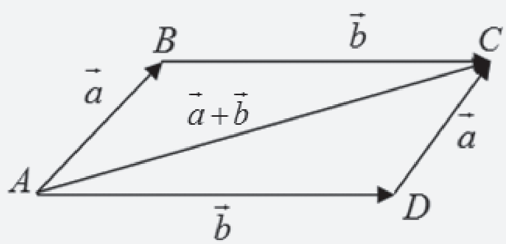
1. $\left(\vec{a}+\vec{b}\right)+\vec{c}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$

   $\vec{a}+\left(\vec{b}+\vec{c}\right)=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$



   $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

   $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AD}+\overrightarrow{DC}$



2. $\vec{b}-\vec{a}, \vec{b}-2\vec{a}, 2\left(\vec{b}-\vec{a}\right), 2\vec{b}-3\vec{a}, \vec{b}-\overrightarrow{2a}$
3. a) Linearly independent.   b) Linearly dependent.
   c) Linearly independent.
4. Linearly independent
5. There are 2^n elements.
6. $\left[\vec{v}\right]_S=\left(c, b-c, a-b\right)$

7. $\left.\begin{array}{l}\vec{u}-\vec{v}=\vec{u}+\left(-\vec{v}\right)\\ k\left(-\vec{v}\right)=-k\vec{v}\end{array}\right\} k\left(\vec{u}-\vec{v}\right)=k\left(\vec{u}+\left(-\vec{v}\right)\right)=k\vec{u}+k\left(-\vec{v}\right)=k\vec{u}-k\vec{v}$

8. $s=-3p+2q+4r$

9. Set

$a_0+a_1t+a_2t^2+a_3t^3=x(1-t)^3+y(1-t)^2+z(1-t)+w.1,\ a_0,a_1,a_2,a_3\in IR$

$$\begin{cases}x=-a_3\\ y=a_2+3a_3\\ z=-a_1-2a_2-3a_3\\ w=a_0+a_1+a_2+a_3\end{cases}$$

Hence the four polynomials generate the space of polynomials of degree ≤ 3.

10. The condition is $2a-4b-3c=0$. Note, in particular, that u, v and w do not generate the whole space $\mathbb{R}^3$.

11. Set $(a,b,0)=xu+yv$ $\begin{cases}x=a\\ y=b-2a\end{cases}$. Hence u and v generate W.

12. a) Set $(0,b,c)=x(0,1,1)+y(0,2,-1)$

$\begin{cases}x=\dfrac{b+2c}{3}\\ y=\dfrac{b-c}{3}\end{cases}$. Hence $(0,1,1)$ and $(0,2,-1)$ generate W.

b) $(0,b,c)=x(0,1,2)+y(0,2,3)+z(0,3,1)$

$\begin{cases}x=-3b+2c+7z\\ y=2b-c-5z\end{cases}$. Hence the three vectors generate W.

13. Any finite set S of polynomials contains one of maximum degree, say n. Then span of S cannot contain polynomials of degree greater than n. Accordingly, $V\neq Span(S)$, for any finite set S.

14. a) dimension is 1 b) dimension is 2

c) dimension is 2 d) dimension is 1

15. a) $\sqrt{213}$ unit of length b) $\sqrt{130}$ unit of length
    c) $\sqrt{26}$ unit of length
16. $\vec{e} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$
17. $k = 7$
18. a) 0 b) $\vec{k}$ c) $-\vec{j}$ d) $\vec{i}$ e) 0 f) 0
19. a) 7 b) 30 c) 15 d) $7\sqrt{3}$
20. a) $(-20, -67, -9)$ b) $(-78, 52, -26)$
    c) $(24, 0, -16)$ d) $(-12, -22, -8)$
    e) $(0, -56, -392)$ d) $(0, 56, 392)$
21. a) $\frac{\sqrt{374}}{2}$ sq. unit b) $9\sqrt{13}$ sq. unit
22. ambiguous, needs parentheses
23. a) 16 cubic unit b) 45 cubic unit
24. a) 9 cubic unit b) $\sqrt{122}$ sq. unit
25. a) $\frac{9\sqrt{2}}{2}$ sq. unit b) $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$ sq. unit
26. $\frac{1}{2}\sqrt{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2}$
27. 6 cubic unit
28. 77.88^0
29. 92
30. $\frac{41}{7}$

Unit 7 Matrices and Determinant of Order 3

Learner's Book page 223 – 264

Aim

Apply matrix and determinant of order 3 to solve related problems.

Objectives

After completing this unit, the learners should be able to:

- Define and give example of matrix of order three.
- Perform different operations on matrices of order three.
- Find the determinant of order three.
- Find the inverse of matrix of order three.
- Solve system of three linear equations by matrix inverse method.

Materials

Exercise books, pens, calculator

Contents

7.1. Square matrices of order 3

Recommended teaching periods: 14 periods

This section looks at the definition of square matrices of order three and their operations:

- Addition and subtraction (only matrices of the same type can be subtracted)

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A-B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

- Transpose

If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $A^t=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

Properties of transpose of matrices

Let A, B be matrices of order three

1. $(A^t)^t = A$
2. $(A + B)^t = A^t + B^t$
3. $(\alpha \times A)^t = \alpha \times A^t$

- **Multiplication**

Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B.

$M_{m\times n} \times M_{n\times p} = M_{m\times p}$

$$A\times B=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\times\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

Properties of multiplication of matrices

Let A, B, C be matrices of order three

1. Associative

$A\times(B\times C)=(A\times B)\times C$

2. Multiplicative Identity

   $A \times I = A$, where *I* is the identity matrix with the same order as matrix *A*.

3. Not Commutative

   $A \times B = B \times A$

4. Distributive

   $A \times (B \times C) = A \times B + A \times C$

5. $(A \times B)^t = B^t \times A^t$

Notice

- If $AB = 0$, it does not necessarily follow that $A = 0$ or $B = 0$.
- **Commuting matrices in multiplication**

  In general, the multiplication of matrices is not commutative, i.e, $AB \neq BA$, but we can have the case where two matrices A and B satisfy $AB = BA$. In this case, A and B are said to be **commuting**.

Teaching guidelines

Let learners know what square matrix of order two is. Square matrix of order three will have three rows and three columns.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in the Learner's Book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 7.1 Page 224

Materials

Exercise book, pens

Answers

$$\begin{pmatrix} 2 & 3 & 0 \\ 1 & -1 & 2 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Exercise 7.1 Page 226

$$\begin{pmatrix} 2 & -3 & 11 \\ 2 & 13 & 1 \\ 0 & 11 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 23 & 24 \\ 1 & 43 & 44 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 12 \\ 1 & -20 & 4 \\ 0 & 18 & 6 \end{pmatrix}, \begin{pmatrix} -4 & 2 & 1 \\ -9 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 9 & 2 \\ 5 & 6 & 4 \\ 7 & 3 & -8 \end{pmatrix}$$

There are many possible answers

Activity 7.2 Page 227

Materials

Exercise book, pens, calculator

Answers

1. $\begin{pmatrix} 5 & 14 & 24 \\ 4 & 21 & 20 \\ 14 & 65 & 12 \end{pmatrix}$ 2. $\begin{pmatrix} 3 & -14 & 20 \\ 1 & -7 & -16 \\ 7 & -17 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Comment: $-A$ is additive inverse of A

4. $\begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix}$ and $\begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix}$

The two results are equal. This implies that the addition of matrices is commutative

5. $\begin{pmatrix} 2 & -4 & 12 \\ 1 & 0 & -4 \\ 5 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 9 & 3 \\ 1 & 9 & 12 \\ 6 & 19 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 15 \\ 2 & 9 & 8 \\ 11 & 21 & 6 \end{pmatrix}$ and

$\begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 3 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 15 \\ 2 & 9 & 8 \\ 11 & 21 & 6 \end{pmatrix}$

The two results are equal. This implies that the addition of matrices is associative

6. $\begin{pmatrix} 2 & 1 & 5 \\ -4 & 0 & 2 \\ 12 & -4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 3 \\ 6 & 7 & 21 \\ 4 & 8 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & -2 \\ -1 & 4 & 0 \end{pmatrix}$

Exercise 7.2 Page 229

1. $\begin{pmatrix} 1 & -10 & -3 \\ -12 & -2 & -11 \\ 0 & -2 & 1 \end{pmatrix}$ 2. $\begin{pmatrix} -25 & 10 & 15 \\ -4 & 6 & -5 \\ -18 & 8 & 25 \end{pmatrix}$ 3. $\begin{pmatrix} 15 & -14 & 3 \\ 0 & 0 & -13 \\ 9 & 3 & 4 \end{pmatrix}$

Activity 7.3 Page 230

Materials

Exercise book, pens, calculator

Answers

1. $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 12 & 3 & -4 \\ 3 & -2 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$ 3. $\begin{pmatrix} 13 & 6 & 0 \\ 4 & -4 & 2 \\ -1 & -3 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$

5. Matrix obtained in 2 is equal to the matrix obtained in 4

6. $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

   Interchanging the rows and columns of matrix A once we get the new matrix $\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$

   Interchanging the rows and columns of matrix A twice we get the new matrix $\begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$

   The last matrix is equal to matrix A

Exercise 7.3 Page 231

1. $$\left(A+B\right)^t=\begin{pmatrix}1 & -3 & 9\\ 6 & 3 & 0\\ 3 & 9 & 13\end{pmatrix}$$

2. $$3A^t+B=\begin{pmatrix}1 & 5 & 10\\ 8 & 9 & -3\\ 12 & 20 & 29\end{pmatrix}$$

3. $$\left(-3B+4A\right)^t=\begin{pmatrix}4 & -19 & 15\\ -4 & -9 & 14\\ -2 & -6 & -4\end{pmatrix}$$

4. $$M^t=\begin{pmatrix}1 & 2 & 4\\ 4 & 1 & 0\\ 1 & 1 & 8\end{pmatrix},\ \begin{pmatrix}1 & 2 & x^2\\ 4 & 1 & 0\\ 1 & x+3 & 8\end{pmatrix}=\begin{pmatrix}1 & 2 & 4\\ 4 & 1 & 0\\ 1 & 1 & 8\end{pmatrix}$$

$$\begin{cases}x^2=4\\ x+3=1\end{cases}\Rightarrow x=-2$$

Activity 7.4 Page 231

Materials

Exercise book, pens, calculator

Answers

$$A\times B=\begin{pmatrix}-2+3-1 & -1+2+6 & 1+3+4\\ 4+6-5 & 2+4+30 & -2+6+20\\ 0+9-4 & 0+6+24 & 0+9+15\end{pmatrix}=\begin{pmatrix}0 & 7 & 8\\ 5 & 36 & 24\\ 5 & 30 & 25\end{pmatrix}$$

Exercise 7.4 Page 233

1. $$A\times B=\begin{pmatrix}-28 & 36 & 39\\ 28 & -6 & -5\\ 56 & 64 & 80\end{pmatrix}$$

2. $$A\times C=\begin{pmatrix}47 & 4 & -36\\ 1 & -9 & 31\\ 112 & 8 & -28\end{pmatrix}$$

3. $$B\times C=\begin{pmatrix}161 & 9 & -21\\ 276 & -22 & -18\\ 123 & 7 & -17\end{pmatrix}$$

Activity 7.5 Page 233

Materials

Exercise book, pens, calculator

Answers

1. $A\times B=\begin{pmatrix}-1 & 3 & -2\\ -2 & -1 & 3\\ 1 & 1 & -2\end{pmatrix}$, $B\times A=\begin{pmatrix}2 & -4 & -1\\ 7 & -7 & -2\\ -5 & 3 & 1\end{pmatrix}$

   $A\times B\neq B\times A$. Multiplication of matrices is not commutative

2. $(A\times B)^t=\begin{pmatrix}-1 & -2 & 1\\ 3 & -1 & 1\\ -2 & 3 & -2\end{pmatrix}$, $B^t\times A^t=\begin{pmatrix}-1 & -2 & 1\\ 3 & -1 & 1\\ -2 & 3 & -2\end{pmatrix}$

   $(A\times B)^t=B^t\times A^t$

3. $A\times(B\times C)\begin{pmatrix}3 & 0 & 1\\ 0 & -1 & 1\\ -2 & 3 & 2\end{pmatrix}\begin{pmatrix}2 & -3 & 1\\ 4 & -3 & 0\\ -2 & 0 & 1\end{pmatrix}=\begin{pmatrix}4 & -9 & 4\\ -6 & 3 & 1\\ 4 & -3 & 0\end{pmatrix}$,

   $(A\times B)\times C=\begin{pmatrix}-1 & 3 & -2\\ -2 & -1 & 3\\ 1 & 1 & -2\end{pmatrix}\begin{pmatrix}1 & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & 0\end{pmatrix}=\begin{pmatrix}4 & -9 & 4\\ -6 & 3 & 1\\ 4 & -3 & 0\end{pmatrix}$

   $A\times(B\times C)=(A\times B)\times C$. Multiplication of matrices is associative

4. $A\times(B+C)=\begin{pmatrix}3 & 0 & 1\\ 0 & -1 & 1\\ -2 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & -2\\ 2 & -1 & -1\\ -2 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 7 & -5\\ -4 & 2 & 2\\ 0 & -5 & 3\end{pmatrix}$

   $A\times B+A\times C=\begin{pmatrix}-1 & 3 & -2\\ -2 & -1 & 3\\ 1 & 1 & -2\end{pmatrix}+\begin{pmatrix}2 & 4 & -3\\ -2 & 3 & -1\\ -1 & -6 & 5\end{pmatrix}=\begin{pmatrix}1 & 7 & -5\\ -4 & 2 & 2\\ 0 & -5 & 3\end{pmatrix}$

   $A\times(B+C)=A\times B+A\times C$. Multiplication of matrices is distributive over addition

Exercise 7.5 Page 237

1. a) $A\times B=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}$, $B\times A=\begin{pmatrix}-2&0&-2\\1&0&1\\-1&0&-1\end{pmatrix}$

b) $(A\times B)\times C=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&-1\\-2&2&0\\-1&1&0\end{pmatrix}=\begin{pmatrix}-5&2&3\\-4&0&4\\0&0&0\end{pmatrix}$

$A\times(B\times C)=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}\begin{pmatrix}-3&1&2\\-1&2&-1\\-2&1&1\end{pmatrix}=\begin{pmatrix}-5&2&3\\-4&0&4\\0&0&0\end{pmatrix}$

c) $A\times(B+C)=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&0\end{pmatrix}\begin{pmatrix}-1&0&0\\-1&2&2\\-2&1&1\end{pmatrix}=\begin{pmatrix}-3&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}$,

$A\times B+A\times C=\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&-1\\2&-1&-1\\0&0&0\end{pmatrix}=\begin{pmatrix}-3&1&1\\-2&-1&-1\\0&0&0\end{pmatrix}$

d) $tr(A\times B)=tr\begin{pmatrix}-3&0&2\\-4&0&0\\0&0&0\end{pmatrix}=-3$

2. $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Activity 7.6 Page 237

Materials

Exercise book, pens, calculator

Answers

1. $(1,0,2)$
2. $(0,1,0)$
3. $(1,-1,0)$
4. $\begin{pmatrix}1&0&1\\0&1&-1\\2&0&0\end{pmatrix}$

Exercise 7.6 Page 241

1. a) $\begin{pmatrix} 3 & 2 & 0 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 4 & -3 & 2 \\ -1 & 0 & 0 \\ -3 & 5 & -1 \end{pmatrix}$

2. a) $\begin{pmatrix} 2 & 1 & -6 \\ 1 & 8 & -15 \\ 11 & -20 & -22 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 9 & 20 \\ -4 & -3 & 4 \\ 10 & 12 & -13 \end{pmatrix}$

c) $\begin{pmatrix} 11 & 4 & -4 \\ 9 & -19 & -10 \\ 11 & -6 & -5 \end{pmatrix}$

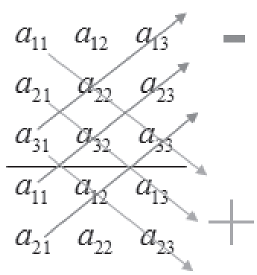
7.2. Determinant of order three

Recommended teaching periods: 17 periods

This section looks at the method used to find the determinant of order three: **Rule of SARRUS**. It looks at the general method used to find determinant of order $n \geq 2$ (cofactor method). It also looks at how to find the inverse of a square matrix of order three .

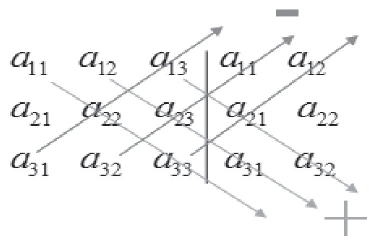
Every linear transformation $f:\mathbb{R}^3 \to \mathbb{R}^3$ can be identified with a matrix of order three, $[f]_{e_j} = (a_{ij})$, whose j^th column is $f(\overrightarrow{e_j})$ where $\{\overrightarrow{e_j}\}$, $j = 1,2,3$ is the standard basis of $\mathbb{R}^3$. The matrix $[f]_{e_j}$ is called matrix representation of f relative to the standard basis $\{\overrightarrow{e_j}\}$.

To calculate the 3x3 determinant, we rewrite the first two rows below the determinant (or first two columns to the right of the determinant).



$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

Or



$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

General rule for $n \times n$ matrices (minor and cofactor)

General method of finding the determinant of matrix with $n \times n$ dimension $(2\times 2,\ 3\times 3,\ 4\times 4,\ 5\times 5,\ldots)$ is the use of cofactors.

Minor

An element, a_{ij}, to the value of the determinant of order $n-1$, obtained by deleting the row i and the column j in the matrix is called a **minor.**

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & [5] & 4 \\ 3 & 6 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

Cofactor

The **cofactor** of the element a_{ij} is its minor prefixing:

The **+** sign if **i+j** is **even.**

The **–** sign if **i+j** is **odd.**

$$\begin{vmatrix} 1 & 2 & 1 \\ [2] & 5 & 4 \\ 3 & 6 & 2 \end{vmatrix} \rightarrow -\begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix}$$

The value of a determinant is equal to the sum of the products of the elements of a line (row or column) by its corresponding cofactors:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Teaching guidelines

Let learners know how to find determinant of order two. For determinant of order three, we have three rows and three columns.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 7.7 Page 242

Materials

Exercise book, pens, calculator

Answers

1. $(1\times 6\times 1)+(3\times 0\times 2)+(5\times 1\times(-4))-(2\times 6\times 5)-(1\times 0\times 1)-(1\times 3\times(-4))=-62$
2. $(10\times 2\times 4)+((-6)\times 5\times 2)+(0\times 3\times 1)-(4\times 5\times 0)-(2\times 3\times 10)-(1\times(-6)\times 2)=-70$

Exercise 7.7 Page 245

1) 82	2) 10	3) -19

Activity 7.8 Page 246

Materials

Exercise book, pens, calculator

Answers

1. $|A| = 0,\ |B| = 0$
2. $|C \cdot D| = -36,\ |C| \cdot |D| = 6 \times (-6) = -36$

   $|C \cdot D| = |C| \cdot |D|$

   Determinant of product is equal to the product of determinants.
3. Product of leading diagonal elements $1 \times 2 \times 3 = 6$,

   $|C| = 6$

   Determinant of a triangular matrix is equal to the product of leading diagonal elements

Exercise 7.8 Page 249

1. $|A| = 0$, $|B| = 0$, $|C| = 14$, $|D| = -5$
2. $|BC| = |B| \times |C| = 0$
3. $|CD| = |C| \times |D| = -70$

Activity 7.9 Page 249

Materials

Exercise book, pens, calculator

Answers

1. $|A| = -1$
2. Cofactor of each element:

   $cofactor(1) = 3,$ $\quad cofactor(1) = -5,$ $\quad cofactor(1) = 1$

   $cofactor(2) = 1,$ $\quad cofactor(1) = -2,$ $\quad cofactor(-1) = 1$

   $cofactor(3) = -2,$ $\quad cofactor(2) = 3,$ $\quad cofactor(1) = -1$

Cofactor matrix

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & -2 & 1 \\ -2 & 3 & -1 \end{pmatrix}$$

3. Transpose of cofactor matrix is $\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$

4. $\dfrac{1}{-1}\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -3+5-1` & -1+2-1 & 2-3+1 \\ -6+5+1 & -2+2-1 & 4-3-1 \\ -9+10+1 & -3+4-1 & 6-6+1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

The product of these two matrices is a unity (identity) matrix I.

Exercise 7.9 Page 251

1. No inverse

2. $\begin{pmatrix} \frac{23}{268} & -\frac{29}{268} & \frac{5}{268} \\ -\frac{3}{268} & -\frac{37}{268} & \frac{11}{268} \\ -\frac{9}{268} & \frac{23}{268} & \frac{33}{268} \end{pmatrix}$

3. $\begin{pmatrix} \frac{6}{7} & -\frac{45}{14} & \frac{16}{7} \\ -\frac{5}{7} & \frac{24}{7} & -\frac{18}{7} \\ \frac{1}{7} & -\frac{11}{14} & \frac{5}{7} \end{pmatrix}$

4. $\begin{pmatrix} -\frac{2}{5} & -\frac{3}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & -1 \\ -\frac{6}{5} & -\frac{29}{5} & 8 \end{pmatrix}$

Activity 7.10 Page 251

Materials

Exercise book, pens, calculator

Answers

1. $$(AB)^{-1}=\begin{pmatrix}0&1&-1\\0&1&0\\1&-3&1\end{pmatrix},\ B^{-1}A^{-1}=\begin{pmatrix}1&-1&0\\1&-1&1\\-2&3&-1\end{pmatrix}\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&1&-1\\0&1&0\\1&-3&1\end{pmatrix}$$

$$(AB)^{-1}=B^{-1}A^{-1}$$

2. $$\left(A^{-1}\right)^{-1}=\begin{pmatrix}1&0&1\\1&-1&1\\0&0&1\end{pmatrix}\qquad \left(A^{-1}\right)^{-1}=A$$

3. $$(4A)^{-1}=\frac{1}{4}\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix},\ \frac{1}{4}A^{-1}=\frac{1}{4}\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}$$

$$(kA)^{-1}=\frac{1}{k}A^{-1},\ k\neq 0$$

4. $$\left(A^{t}\right)^{-1}=\begin{pmatrix}1&1&0\\0&-1&0\\-1&0&1\end{pmatrix},\ \left(A^{-1}\right)^{t}=\begin{pmatrix}1&0&-1\\1&-1&0\\0&0&1\end{pmatrix}^{t}=\begin{pmatrix}1&1&0\\0&-1&0\\-1&0&1\end{pmatrix}$$

$$\left(A^{t}\right)^{-1}=\left(A^{-1}\right)^{t}$$

Exercise 7.10 Page 253

1. $$A^{-1}=\frac{1}{11}\begin{pmatrix}2&5&-14\\1&-3&4\\-1&3&7\end{pmatrix},\ B^{-1}=\begin{pmatrix}-\frac{5}{12}&\frac{1}{4}&-\frac{13}{12}\\-\frac{3}{4}&\frac{1}{4}&-\frac{3}{4}\\\frac{1}{3}&0&\frac{2}{3}\end{pmatrix}$$

2. $$\left(A^{-1}\right)^{-1}=\begin{pmatrix}3&7&2\\1&0&2\\0&1&1\end{pmatrix}$$

3. $(10A)^{-1} = \frac{1}{110}\begin{pmatrix} 2 & 5 & -14 \\ 1 & -3 & 4 \\ -1 & 3 & 7 \end{pmatrix}$

4. $(A^t)^{-1} = \frac{1}{11}\begin{pmatrix} 2 & 1 & -1 \\ 5 & -3 & 3 \\ -14 & 4 & 7 \end{pmatrix}$

7.3. Applications

Recommended teaching periods: 4 periods

This section looks at how to solve a system of three linear equations by matrix inverse and by Cramer's rule method.

Consider the following simultaneous linear equations:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

One of the methods of solving this, is the use Cramer's rule.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \qquad \Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

Remember that if $\Delta = 0$, there is no solution or infinity number of solution

Teaching guidelines

Let learners know how to rewrite a system of linear equation in matrix form, how to find inverse of matrix and how to multiply to matrices.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 7.11 Page 253

Materials

Exercise book, pens

Answers

1. $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

2. $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Exercise 7.11 Page 257

1. $S=\{(0,0,0)\}$ 2. $S=\{\ \}$ 3. $S=\{(1,2,0)\}$

End of Unit Assessment Page 262

1. a) $\begin{pmatrix} -7 & -3 & 0 \\ 0 & 4 & -12 \\ 0 & -10 & -2 \end{pmatrix}$ b) $\begin{pmatrix} -9 & -23 & 6 \\ 0 & -4 & -16 \\ -4 & 2 & -8 \end{pmatrix}$

c) $\begin{pmatrix} 7 & 8 & 3 \\ 2 & 4 & -10 \\ 2 & -14 & 7 \end{pmatrix}$ d) $\begin{pmatrix} 29 & 28 & 15 \\ 10 & -34 & -21 \\ -4 & 28 & -16 \end{pmatrix}$

e) $\begin{pmatrix} 38 & 36 & 13 \\ -1 & 12 & -42 \\ 0 & 0 & -18 \end{pmatrix}$ f) $\begin{pmatrix} 118 & 120 & 37 \\ 19 & 12 & 10 \\ 14 & 0 & 76 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 4 & 0 \end{pmatrix}$

3. a) $[f]_e = \begin{pmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{pmatrix}$

b) Let $\vec{v}=(a,b,c)\in\mathbb{R}^3$ $\quad [\vec{v}]_e = \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix}$

$$[f]_e[\vec{v}]_e = \begin{pmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{pmatrix}\begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix} = \begin{pmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{pmatrix}$$

But, $f(\vec{v}) = f(a,b,c) = (2b+c, a-4b, 3a)$,

$[f(\vec{v})]_e = \begin{pmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{pmatrix}$, verified.

4. a) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ b) no inverse c) $\begin{pmatrix} \frac{44}{207} & -\frac{8}{207} & \frac{1}{69} \\ \frac{1}{207} & -\frac{19}{207} & \frac{11}{69} \\ -\frac{13}{207} & \frac{40}{207} & -\frac{5}{69} \end{pmatrix}$

5. $X = \begin{pmatrix} 3 & -2 & -2 \\ -5 & 5 & 2 \\ 5 & -3 & 1 \end{pmatrix}$

6. a) $S = \{(0,0,0)\}$ b) $S = \{(1,1,1)\}$

   c) $S = \{(3,0,1)\}$

7. 0

8. $k \neq -\frac{3}{5}$, $A^{-1} = \frac{1}{8}\begin{pmatrix} -29 & 17 & 14 \\ -9 & 5 & 6 \\ 16 & -8 & -8 \end{pmatrix}$

9. a) $A^{-1} = \frac{1}{7}\left(4I - A^2\right)$ b) $\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

10. $A^3 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I$

    a) $x = -3, y = 5, z = 2$ b) $x = 2, y = 1, z = 0$

11. $-t^3 + t^2 + t - 1$

12. a) x^2 b) x^n c) $2x$ d) mx

13. $k = \frac{1}{4}$

14. a) $\lambda = 5, \mu \neq 9$ b) $\lambda \neq 5$ c) $\lambda = 5, \mu = 9$

Unit 8 Points, Straight Lines, Planes and Sphere in 3D

Learner's Book page 265 – 404

Aim

Use algebraic representations of points, lines, spheres and planes in 3D space and solve related problems

Objectives

After completing this unit, the learners should be able to:

- plot points in three dimensions.
- find equations of straight lines in three dimensions.
- find equations of planes in three dimensions.
- find equations of sphere in three dimensions.

Materials

Exercise books, pens, instruments of geometry, calculator

Contents

8.1. Points in three dimensions

Recommended teaching periods: 7 periods

This section looks at the method used to locate a point in space.

It also looks at;

- Midpoint of a segment,

  Let the points (x_1, y_1, z_1) and (x_2, y_2, z_2) be the endpoints of a line segment, then the midpoint of that segment is given by the formula:

  $$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

- Centroid of a geometric figure

The **centroid** of **geometric figure (barycentre or geometric centre)** is the arithmetic mean (average) position of all points in the shape. In physics, barycentre means the **physical centre** of mass or the **centre of gravity**.

Let (x_1, y_1, z_1), (x_2, y_2, z_2),...., (x_n, y_n, z_n) be n points of space, their centroid is given by the formula:

$$\left(\frac{x_1 + x_2 + ... + x_n}{n}, \frac{y_1 + y_2 + ... + y_n}{n}, \frac{z_1 + z_2 + ... + z_n}{n}\right)$$

Ratio formula:

If P is a point on the line AB such that P divides AB internally in the ratio $m:n$, then $P = \frac{mB + nA}{m+n}$ and if P divides AB externally in the ratio $m:n$, then $P = \frac{mB - nA}{m-n}$.

Teaching guidelines

Let learners know how to plot points in 2-dimensions and how to find a midpoint of two points in 2-dimension.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in the Learner's Book or through your own examples.

- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 8.1 Page 266

Materials

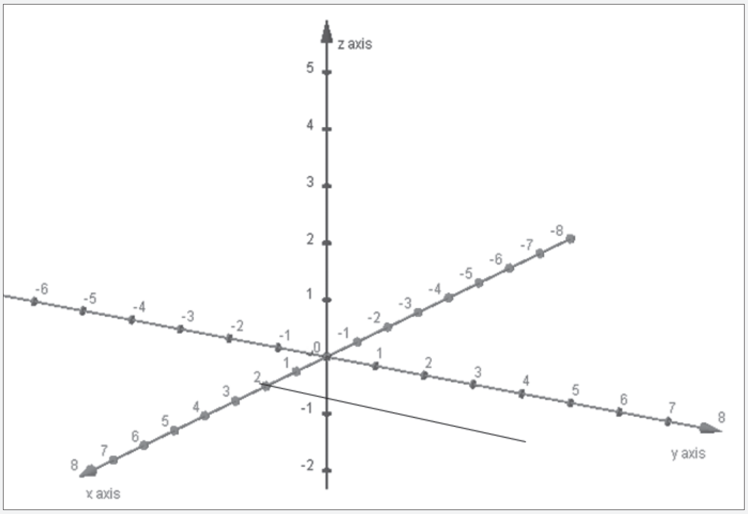
Exercise book, pens, instruments of geometry

Answers

Suppose that we need to represent the point $A(2,3,5)$ in space

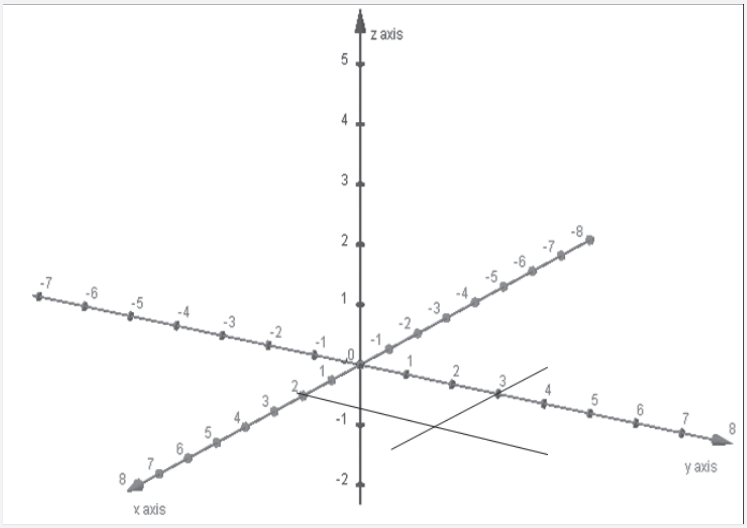
- From x-coordinate 2, draw a line parallel to y-axis

  See following figure.



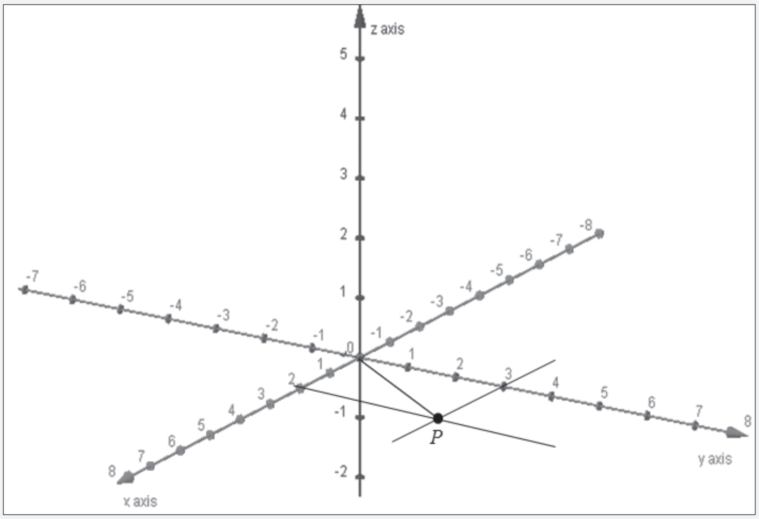
- From *y-coordinate* 3, draw another line parallel to *x-axis*

  See following figure.



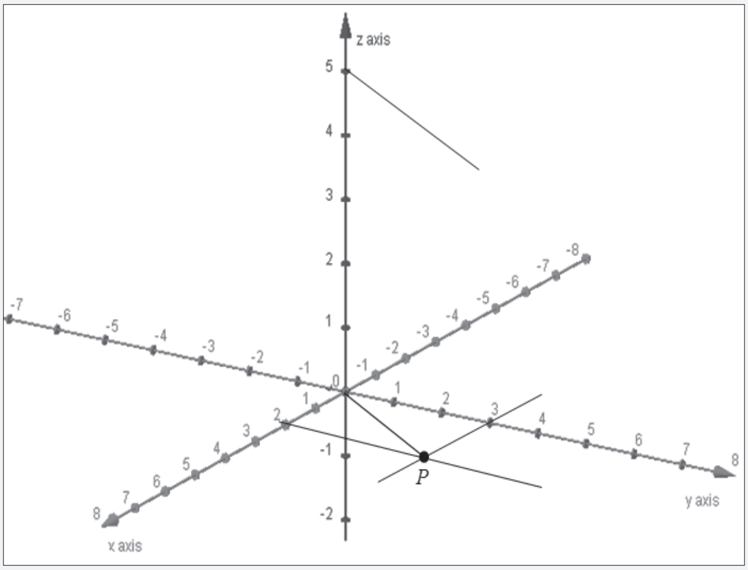
- Now you have a point of intersection of two lines, let call it *P*. From this point, draw another line parallel to *z-axis* and another joining this point and origin of coordinates which is line $0P$

  See following figure.



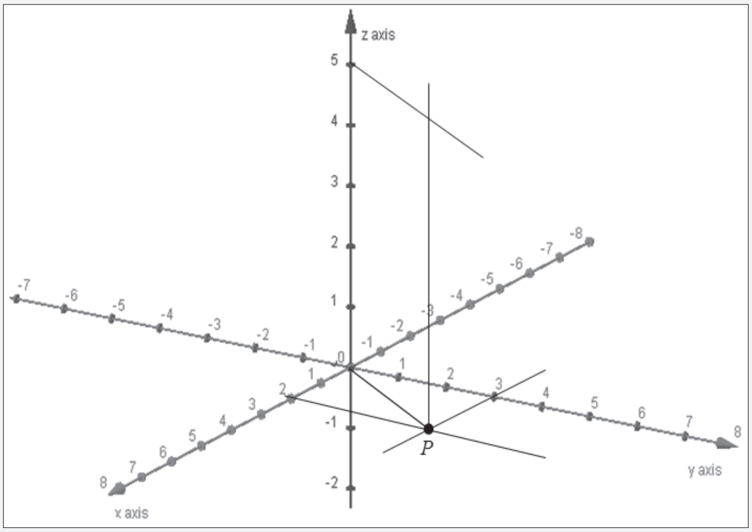
- From *z-coordinate*, 5, draw another line parallel to the line $0P$

See following figure.



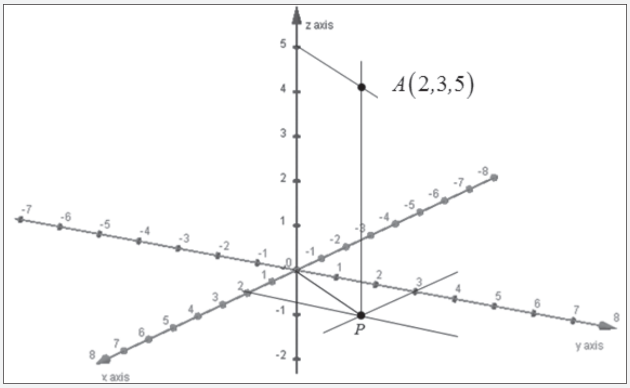
- Draw another line parallel to *z-axis* and passing through point *P*

See following figure.



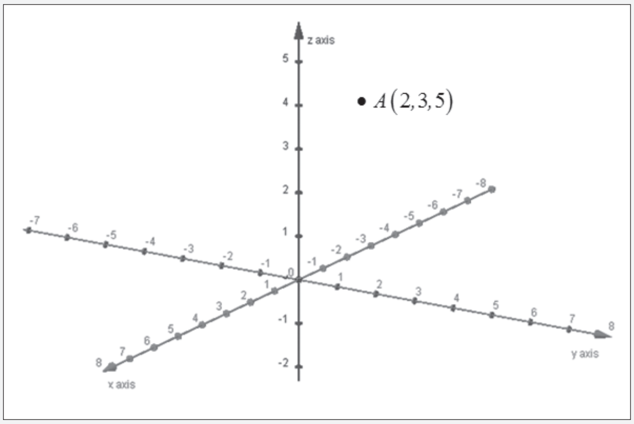
- Now, you have the intersection of the line from *z-coordinate* and the line parallel to *z-axis*. this intersection is the needed point $A(2,3,5)$

See following figure.

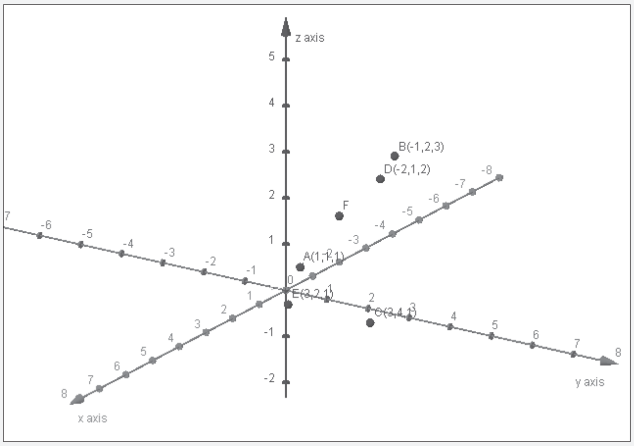


- Clean unwanted points and lines to remain with the needed point.

Finally, we have



Exercise 8.1 Page 268



Activity 8.2 Page 268

Materials

Exercise book, pens, calculator

Answers

1. $\left(\frac{3}{2},\frac{5}{2},3\right)$ 2. $\left(-2,\frac{5}{3},\frac{1}{3}\right)$

Exercise 8.2 Page 271

1. a) $\left(0,\frac{7}{2},\frac{11}{2}\right)$ b) $\left(6,\frac{5}{2},-\frac{1}{2}\right)$

   c) $\left(-\frac{7}{2},\frac{11}{2},5\right)$ d) $\left(1,0,\frac{9}{4}\right)$

2. a) $\left(\frac{8}{3},2,\frac{7}{3}\right)$ b) $\left(2,-\frac{1}{3},6\right)$

   c) $\left(-\frac{1}{2},\frac{15}{4},7\right)$ d) $\left(3,1,\frac{1}{2}\right)$

Activity 8.3 Page 271

Materials

Exercise book, pens

Answers

1.

$$\overrightarrow{AP}=\frac{m}{n}\overrightarrow{PB}$$

$$\Leftrightarrow P-A=\frac{m}{n}(B-P) \quad \Leftrightarrow P-A=\frac{m}{n}B-\frac{m}{n}P$$

$$\Leftrightarrow P+\frac{m}{n}P=\frac{m}{n}B+A \quad \Leftrightarrow nP+mP=mB+nA$$

$$\Leftrightarrow P(n+m)=mB+nA \Rightarrow P=\frac{mB+nA}{n+m}$$

2.

$$\overrightarrow{PA}=\frac{m}{n}\overrightarrow{PB}$$

$$\Leftrightarrow A-P=\frac{m}{n}(B-P) \quad \Leftrightarrow A-P=\frac{m}{n}B-\frac{m}{n}P$$

$$\Leftrightarrow \frac{m}{n}P-P=\frac{m}{n}B-A \quad \Leftrightarrow mP-nP=mB-nA$$

$$\Leftrightarrow P(m-n)=mB-nA \Rightarrow P=\frac{mB-nA}{m-n}$$

Or

$$\overrightarrow{AP}=\frac{m}{n}\overrightarrow{BP}$$

$$\Leftrightarrow P-A=\frac{m}{n}(P-B) \quad \Leftrightarrow P-A=\frac{m}{n}P-\frac{m}{n}B$$

$$\Leftrightarrow P-\frac{m}{n}P=A-\frac{m}{n}B \quad \Leftrightarrow nP-mP=nA-mB$$

$$\Leftrightarrow P(n-m)=nA-mB$$

$$P=\frac{nA-mB}{n-m}$$

$$=\frac{-(mB-nA)}{-(m-n)}$$

$$=\frac{mB-nA}{m-n}$$

Exercise 8.3 Page 272

1. Internally: $P=\frac{2B+3A}{5}=\frac{1}{5}(2B+3A)=\frac{1}{5}(14,9,29)$

   Externally, $P=\frac{2B-3A}{-1}=3A-2B=(-2,-3,1)$

2. $\left(\frac{13}{5},\frac{8}{5},\frac{14}{5}\right)$

3.	i) 2:3 ii) -2:3	4.	i) 2:3 ii) 2:3
5.	$\frac{1}{2}:1$ or 1:2	6.	$x=9,\ z=5$

8.2. Straight lines in three dimensions

Recommended teaching periods: 9 periods

This section looks

a) Equations of line in space.

A line parallel to the vector $\vec{v}=(a,b,c)$ and passing through the point P with position vector $\overrightarrow{0P}=(x_0,y_0,z_0)$ has:

Vector equation

$\overrightarrow{\mathrm{OQ}}=\overrightarrow{\mathrm{OP}}+\mathrm{r}\vec{\mathrm{v}}$ or $(x,y,z)=(x_0,y_0,z_0)+r(a,b,c)$

or $x\vec{i}+y\vec{j}+z\vec{k}=x_0\vec{i}+y_0\vec{j}+z_0\vec{k}+r\left(a\vec{i}+b\vec{j}+c\vec{k}\right)$, r is a parameter.

Parametric equations

$$\begin{cases} x = x_0 + ra \\ y = y_0 + rb \\ z = z_0 + rc \end{cases}$$

Cartesian equations (or symmetric equations)

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Remember this:

If a given line is parallel to the vector $\vec{r}=(a,b,c)$, $\vec{r}$ is called its direction vector.

For the line passing through points $P(x_0,y_0,z_0)$ and $Q(x_1,y_1,z_1)$, with $V(x,y,z)$ any point on the line , has

Vector equation

$\overrightarrow{PV} = r\overrightarrow{PQ}$, where r is a parameter.

Parametric equations:

$$\begin{cases} x = x_0 + r(x_1 - x_0) \\ y = y_0 + r(y_1 - y_0) \\ z = z_0 + r(z_1 - z_0) \end{cases}$$

Symmetric equations:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Here, the direction vector is $\overrightarrow{PQ}$.

b) Condition of co-linearity of three points

The three points (a_1, a_2, a_3) ; (b_1, b_2, b_3) and (c_1, c_2, c_3) are collinear (means that they lie on the same line) if the following conditions are satisfied

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

c) Relation between two lines

Two lines which are not parallel and do not intersect, are called skew lines.

Two lines are parallel when their direction vectors are proportional.

d) Angle between two lines

The angle between two lines is equal to the angle between their direction vectors.

Let $\vec{u}$ and $\vec{v}$ be direction vectors of two lines l_1 and l_2 respectively,

θ angle between l_1 and l_2, thus, $\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$

e) Distance from a point to a line

The distance from point $B(b_1, b_2, b_3)$ to the line passing through point $A(a_1, a_2, a_3)$ with direction vector $\vec{u} = (c_1, c_2, c_3)$ is $\frac{\left\|\overrightarrow{AB} \times \vec{u}\right\|}{\left\|\vec{u}\right\|}$.

Distance between two skew lines

Consider two skew lines $L_1 : \vec{r} = \vec{a} + \lambda \vec{u}$ and $L_2 : \vec{r} = \vec{b} + \lambda \vec{v}$.

The shortest distance between these lines is

$$d = \frac{\left\|\overrightarrow{ab} \cdot \vec{u} \times \vec{v}\right\|}{\left\|\vec{u} \times \vec{v}\right\|} \text{ or } d = \frac{\left[\overrightarrow{ab} \quad \vec{u} \quad \vec{v}\right]}{\left\|\vec{u} \times \vec{v}\right\|}$$

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 8.4 Page 273

Materials

Exercise book, pens

Answers

1. $(x,y,z)=(x_0,y_0,z_0)+r(a,b,c)$

$$x\vec{i}+y\vec{j}+z\vec{k}=x_0\vec{i}+y_0\vec{j}+z_0\vec{k}+r\left(a\vec{i}+b\vec{j}+c\vec{k}\right)$$

Equate the respective components, there are three equations

$$\begin{cases} x=x_0+ra \\ y=y_0+rb \\ z=z_0+rc \end{cases}$$

2. Eliminating parameter r, we have

$$\begin{cases} r=\dfrac{x-x_0}{a} \\ r=\dfrac{y-y_0}{b} \\ r=\dfrac{z-z_0}{c} \end{cases} \quad \text{Or} \quad \frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Exercise 8.4 Page 277

1. $x\vec{i}+y\vec{j}+z\vec{k}=\vec{i}+\vec{j}+\vec{k}+\lambda\left(2\vec{i}+\vec{j}+3\vec{k}\right)$

$$\begin{cases} x=1+2\lambda \\ y=1+\lambda \\ z=1+3\lambda \end{cases} \qquad \frac{x-1}{2}=\frac{y-1}{1}=\frac{z-1}{3}$$

2. $x\vec{i}+y\vec{j}+z\vec{k}=-2\vec{i}+3\vec{j}+\vec{k}+\lambda\left(2\vec{i}+\vec{j}+3\vec{k}\right)$

$$\begin{cases} x = -2 + 2\lambda \\ y = 3 + \lambda \\ z = 1 + 3\lambda \end{cases} \qquad \frac{x+2}{2} = \frac{y-3}{1} = \frac{z-1}{3}$$

3. $x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = 9\vec{i} + 3\vec{j} + \lambda\left(\vec{i} + \vec{j} + 6\vec{k}\right)$

$$\begin{cases} x = 9 + \lambda \\ y = 3 + \lambda \\ z = 6\lambda \end{cases} \qquad \frac{x-9}{1} = \frac{y-3}{1} = \frac{z}{6}$$

4. $x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = 4\vec{i} + 5\vec{j} + 2\vec{k} + \lambda\left(-3\vec{i} + 2\vec{j} + \vec{k}\right)$

$$\begin{cases} x = 4 - 3\lambda \\ y = 5 + 2\lambda \\ z = 2 + \lambda \end{cases} \qquad \frac{x-4}{-3} = \frac{y-5}{2} = \frac{z-2}{1}$$

Activity 8.5 Page 278

Materials

Exercise book, pens

Answers

1. The direction vector is

$\overrightarrow{PQ} = (x_1, y_1, z_1) - (x_0, y_0, z_0) = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

Now the vector equation is given by

$\overrightarrow{PV} = r\overrightarrow{PQ}$ or $\overrightarrow{OV} = \overrightarrow{OP} + r\overrightarrow{PQ}$,

With $V(x, y, z)$, $0(0,0,0)$ and r is a parameter.

2. Parametric equations:

$$\begin{cases} x = x_0 + r(x_1 - x_0) \\ y = y_0 + r(y_1 - y_0) \\ z = z_0 + r(z_1 - z_0) \end{cases}$$

3. Eliminating the parameter, we have the symmetric equations:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Exercise 8.5 Page 281

1. $x\vec{i}+y\vec{j}+z\vec{k}=2\vec{i}+\vec{j}+4\vec{k}+\lambda\left(\vec{i}-3\vec{k}\right)$

$$\begin{cases} x=2+\lambda \\ y=1 \\ z=4-3\lambda \end{cases}$$

$$\frac{x-2}{1}=\frac{y-1}{0}=\frac{z-4}{-3} \text{ or } \frac{x-2}{1}=\frac{z-4}{-3},\ y=1$$

2. $x\vec{i}+y\vec{j}+z\vec{k}=\vec{i}+\vec{j}+3\vec{k}+\lambda\left(\vec{i}+4\vec{j}+\vec{k}\right)$

$$\begin{cases} x=1+\lambda \\ y=1+4\lambda \\ z=3+\lambda \end{cases} \qquad \frac{x-1}{1}=\frac{y-1}{4}=\frac{z-3}{1}$$

3. $x\ \vec{i}+y\ \vec{j}+z\ \vec{k}=2\vec{i}+\vec{j}+4\vec{k}+\lambda\left(4\vec{i}+2\vec{j}-2\vec{k}\right)$

$$\begin{cases} x=2+4\lambda \\ y=1+2\lambda \\ z=4-2\lambda \end{cases} \qquad \frac{x-2}{4}=\frac{y-1}{2}=\frac{z-4}{-2}$$

4. $x\vec{i}+y\vec{j}+z\vec{k}=\vec{i}+\vec{j}+\vec{k}+\lambda\left(3\vec{i}+4\vec{j}+5\vec{k}\right)$

$$\begin{cases} x=1+3\lambda \\ y=1+4\lambda \\ z=1+5\lambda \end{cases} \qquad \frac{x-1}{3}=\frac{y-1}{4}=\frac{z-1}{5}$$

Activity 8.6 Page 281

Materials

Exercise book, pens, calculator

Answers

A) Equation of line passing through points $(1,2,3)$ and $(1,-4,3)$ is

$$\begin{vmatrix} x & 1 & 1 \\ y & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & 1 & 1 \\ z & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

If point $(-1,0,5)$ lies on this line, then

$$\begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = -2+0-4-2-4-0=-12$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -3+5+3+3-3-5=0$$

Since $\begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = -12 \neq 0$, the given three points do not lie on the same line.

B) Equation of line passing through points $(3,4,7)$ and $(5,-2,1)$ is

$$\begin{vmatrix} x & 3 & 5 \\ y & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & 3 & 5 \\ z & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

If point $(4,1,4)$ lies on this line, then

$$\begin{vmatrix} 4 & 3 & 5 \\ 1 & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 5 \\ 4 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 3 & 5 \\ 1 & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 16+5-6-20+8-3=0$$

$$\begin{vmatrix} 4 & 3 & 5 \\ 4 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 28+20+3-35-4-12=0$$

Since $\begin{vmatrix} 4 & 3 & 5 \\ 1 & 4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 5 \\ 4 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$, the given three points lie on the same line.

C) Equation of line passing through points $(1,9,3)$ and $(1,8,5)$ is

$$\begin{vmatrix} x & 1 & 1 \\ y & 9 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & 1 & 1 \\ z & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

If point $(1,10,1)$ lies on this line, then

$$\begin{vmatrix} 1 & 1 & 1 \\ 10 & 9 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 3+5+1-3-5-1=0$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -3+5+3+3-3-5=0$$

Since $\begin{vmatrix} 1 & 1 & 1 \\ 10 & 9 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = 0$, the given three points lie on the same line.

The three points

(a_1, a_2, a_3) ; (b_1, b_2, b_3) and (c_1, c_2, c_3) are lie on the same line if the following conditions are satisfied

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Exercise 8.6 Page 283

1. Not collinear
2. $x=1+2t,\ y=1+t,\ z=1+3t$
3. $a=7$

Activity 8.7 Page 283

Materials

Exercise book, pens, calculator

Answers

1. $L_1: x=2-\lambda,\ y=2+2\lambda,\ z=1+3\lambda$,
   $L_2: x=1-\mu,\ y=1+2\mu,\ z=1+3\mu$

   L_1 and L_2 have the same direction vectors, thus L_1 and L_2 are parallel.

   Let us check if they are coincident:

   $$\begin{cases}2-\lambda=1-\mu\\2+2\lambda=1+2\mu\\1+3\lambda=1+3\mu\end{cases}$$

   $$\Rightarrow\begin{cases}\lambda=1+\mu & (1)\\2+2\lambda=1+2\mu & (2)\\1+3\lambda=1+3\mu\Rightarrow\lambda=\mu & (3)\end{cases}$$

   From (1) and (3), we find that there is no solution.

   Therefore, L_1 and L_2 are different.

2. $L_1: x=\lambda,\ y=-2+2\lambda,\ z=5-\lambda$,
   $L_2: x=1-\mu,\ y=-3-3\mu,\ z=4+\mu$

   Direction vectors of L_1 and L_2 are not proportional, thus L_1 and L_2 are not parallel.

   Let us find their intersection:

   $$\begin{cases}\lambda=1-\mu\\-2+2\lambda=-3-3\mu\\5-\lambda=4+\mu\end{cases}$$

$$\Rightarrow\begin{cases}\lambda=1-\mu\\-2+2(1-\mu)=-3-3\mu\\5-(1-\mu)=4+\mu\end{cases}\Leftrightarrow\begin{cases}\lambda=1-\mu\\-2+2-2\mu=-3-3\mu\\5-1+\mu=4+\mu\end{cases}$$

$$\Leftrightarrow\begin{cases}\lambda=1-\mu\\-2+2-2\mu=-3-3\mu\Rightarrow\mu=-3\\5-1+\mu=4+\mu\Rightarrow 0\mu=0\end{cases}$$

$$\begin{cases}\mu=-3\\\lambda=4\end{cases}$$

For $\lambda=4$, $x=4,\ y=6,\ z=1$

Intersection point is $(4,6,1)$

3. $L_1: x=5+2\lambda,\ y=4+\lambda,\ z=5+\lambda$,
$L_2: x=1+2\mu,\ y=2+\mu,\ z=3+\mu$

Direction vectors of L_1 and L_2 are proportional, thus L_1 and L_2 are parallel.

Let us check if they are coincident:

$$\begin{cases}5+2\lambda=1+2\mu\\4+\lambda=2+\mu\\5+\lambda=3+\mu\end{cases}\Rightarrow\begin{cases}\lambda=-2+\mu\\4+2(-2+\mu)=2+\mu\\5+(-2+\mu)=3+\mu\end{cases}$$

$$\Leftrightarrow\begin{cases}\lambda=2+\mu\\4-4+2\mu=2+\mu\\5-2+\mu=3+\mu\end{cases}\Leftrightarrow\begin{cases}\lambda=2+\mu\\4-4+2\mu=2+\mu\Rightarrow\mu=2\\5-2+\mu=3+\mu\Rightarrow 0\mu=0\end{cases}$$

From the values of there are many solutions. The two lines coincide.

4. $L_1: \begin{cases}x=2+8s\\y=4-3s\\z=5+s\end{cases}\quad L_2: \begin{cases}x=1+4t\\y=5-4t\\z=-1+5t\end{cases}$

Direction vectors of L_1 and L_2 are not proportional, thus L_1 and L_2 are not parallel.

Let us find their intersection:

$$\begin{cases} 2+8s=1+4t \\ 4-3s=5-4t \\ 5+s=-1+5t \end{cases} \Rightarrow \begin{cases} s=\dfrac{-1+4t}{8} \\ 4-3\dfrac{-1+4t}{8}=5-4t \\ 5+\dfrac{-1+4t}{8}=-1+5t \end{cases}$$

$$\Leftrightarrow \begin{cases} s=\dfrac{-1+4t}{8} \\ 32+3-12t=40-32t \Rightarrow 20t=5 \\ 40-1+4t=-8+40t \Rightarrow 36t=47 \end{cases}$$

There is no solution for these simultaneous equations.

Therefore L_1 and L_2 do not intersect.

Exercise 8.7 Page 290

1.	Skew	2.	Intersect at (2,1,-7)
3.	Skew	4.	Skew
5.	Skew	6.	Intersect at (1,1,1)

Activity 8.8 Page 290

Materials

Exercise book, pens, calculator

Answers

1)	39^0	2)	39^0 , 141^0

Exercise 8.8 Page 292

1)	$\dfrac{\pi}{4}$	2)	1.76 radians	3)	0.82 radians	4)	79^0
5)	80.41^0	6)	48.70^0	7)	68.48^0		

Activity 8.9 Page 293

Materials

Exercise book, pens

Answers

1. $\sin\theta = \frac{d}{\left\|\overrightarrow{AB}\right\|} \Rightarrow d = \left\|\overrightarrow{AB}\right\| \sin\theta$

2. $d = \frac{\left\|\overrightarrow{AB}\right\| \left\|\vec{u}\right\| \sin\theta}{\left\|\vec{u}\right\|}$ and $d = \frac{\left\|\overrightarrow{AB} \times \vec{u}\right\|}{\left\|\vec{u}\right\|}$

Exercise 8.9 Page 294

1) $2\sqrt{30}$ unit of length

2) 0 unit of length

3) $\frac{9\sqrt{42}}{7}$ unit of length

4) $\frac{3\sqrt{10}}{2}$ unit of length

5) $\frac{\sqrt{2}}{3}$ unit of length

Activity 8.10 Page 295

Materials

Exercise book, pens, calculator

Answers

1. Direction vectors of the lines are $(3,4,-2)$ and $(6,-4,-1)$

   Since the two direction vectors are not proportional, the lines are not parallel.

   Check if there is a common point:

$$\begin{cases} -7+3t = 21+6s \\ -4+4t = -5+4s \\ -3-2t = 2+s \end{cases} \qquad \begin{cases} t = \frac{28+6s}{3} \\ -4+4\left(\frac{28+6s}{3}\right) = -5+4s \\ -3-2\left(\frac{28+6s}{3}\right) = 2+s \end{cases}$$

$$\begin{cases} t = \dfrac{28+6s}{3} \\ -12+112+24s = -15+12s \\ -9-56-12s = 6+3s \end{cases} \quad \begin{cases} 12s = -115 \\ -15s = 71 \end{cases} \text{ impossible.}$$

The lines are skew.

2. Let (a,b,c) be the perpendicular vector to both lines

$$\begin{cases} 3a+4b-2c=0 \\ 6a-4b-c=0 \Rightarrow c=6a-4b \end{cases}$$

$$3a+4b-12a+8b=0$$

$$\Leftrightarrow 12b=9a \Rightarrow b=\frac{3a}{4}$$

Let $a=1$, $b=\dfrac{3}{4}$ and $c=6-3=3$.

The perpendicular vector is $\left(1,\dfrac{3}{4},3\right)$ or $(4,3,12)$

The normalized vector of this vector is

$$\frac{1}{\sqrt{169}}(4,3,12)=\left(\frac{4}{13},\frac{3}{13},\frac{12}{13}\right)$$

3. Point on first line is $(-7,-4,-3)$, point on second line is $(21,-5,2)$. Vector joining these points $(28,\ \ 1,5)$

4. The needed scalar product is

$$\left(\frac{4}{13},\frac{3}{13},\frac{12}{13}\right)\cdot(28,-1,5)=13$$

Exercise 8.10 Page 301

1. $\dfrac{95\sqrt{1817}}{1817}$ unit of length 2. 0 unit of length

3. $\dfrac{68\sqrt{230}}{115}$ unit of length

8.3. Planes in three dimensions

Recommended teaching periods: 9 periods

This section looks at:

a) Equations of plane in space.

Equations of plane containing point $P(x_0, y_0, z_0)$, with direction vector $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_3)$ and $X(x, y, z)$ any point on this plane are the following:

Vector equation

$\overrightarrow{PX} = r\vec{u} + s\vec{v}$ where r, s are parameters.

Parametric equations

$$\begin{cases} x = x_0 + rx_1 + sx_2 \\ y = y_0 + ry_1 + sy_2 \\ z = z_0 + rz_1 + sz_2 \end{cases}$$

Cartesian equation

$$\begin{vmatrix} x - x_0 & x_1 & x_2 \\ y - y_0 & y_1 & y_2 \\ z - z_0 & z_1 & z_2 \end{vmatrix} = 0$$

We can also find the Cartesian equation by the following determinant:

$$\begin{vmatrix} x & x_0 & x_1 & x_2 \\ y & y_0 & y_1 & y_2 \\ z & z_0 & z_1 & z_2 \\ 1 & 1 & 0 & 0 \end{vmatrix} = 0$$

The Cartesian equation of plane can be written in the form $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

If $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ are two points of a plane whose direction vector are $\vec{v} = (x_2, y_2, z_2)$ and $X(x, y, z)$ any point on this plane , its equations are given as follows:

Vector equation

$\overrightarrow{PX} = r\overrightarrow{PQ} + s\vec{v}$ where r and s are parameters

Parametric equations

$$\begin{cases} x = x_0 + r(x_1 - x_0) + sx_2 \\ y = y_0 + r(y_1 - y_0) + sy_2 \\ z = z_0 + r(z_1 - z_0) + sz_2 \end{cases}$$

Cartesian equation

$$\begin{vmatrix} x - x_0 & x_1 - x_0 & x_2 \\ y - y_0 & y_1 - y_0 & y_2 \\ z - z_0 & z_1 - z_0 & z_2 \end{vmatrix} = 0$$

Or we can use the determinant

$$\begin{vmatrix} x & x_0 & x_1 & x_2 \\ y & y_0 & y_1 & y_2 \\ z & z_0 & z_1 & z_2 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

b) Condition of co-planarity of four points

Four points

(a_1, a_2, a_3) ; (b_1, b_2, b_3) ;(c_1, c_2, c_3) *and* (d_1, d_2, d_3)

are coplanar (means that they lie on the same plane) if the following condition is satisfied.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a_1 - d_1 & b_1 - d_1 & c_1 - d_1 \\ a_2 - d_2 & b_2 - d_2 & c_2 - d_2 \\ a_3 - d_3 & b_3 - d_3 & c_3 - d_3 \end{vmatrix} = 0$$

c) Position of a line and a plane

A line L is perpendicular to plane α if and only if each direction vector of L is perpendicular to each direction vector of α.

A line and plane are parallel if the direction vector of the line is perpendicular to the normal vector of the plane:

two possibilities occur:

a line and plane are strictly parallel or a line lies in the plane

d) Angle between lines and planes

Angle between a line and plane

The angle which line L makes with plane α is defined to be the angle θ which is complement of angle between the direction vector of L and normal to the plane α.

Thus, $\theta = 90^0 - \arccos\left(\frac{\vec{n}\cdot\vec{u}}{|\vec{n}|\cdot|\vec{u}|}\right)$ or $\theta = \arcsin\left(\frac{\vec{n}\cdot\vec{u}}{\|\vec{n}\|\cdot\|\vec{u}\|}\right)$

Angle between two planes

The angle θ between planes α and β is defined to be an angle between their normal vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ respectively.

Thus, $\theta = \arccos\left(\frac{\overrightarrow{n_1}\cdot\overrightarrow{n_2}}{|\overrightarrow{n_1}|\cdot|\overrightarrow{n_2}|}\right)$

e) Distance between lines and planes

Distance from a point to the plane

The distance from point $B(b_1, b_2, b_3)$ to the plane $\alpha \equiv ax + by + cz = d$ is

$$d(B,\alpha) = \frac{|ab_1 + bb_2 + cb_3 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between two planes

When calculating the distance between two planes, first check if the planes are parallel. If they are not, they intersect and the distance is zero. If they are parallel, find an arbitrary point in one of the planes and calculate its distance to the other plane.

Note that if two planes coincide (identical), the shortest distance is zero.

Shortest distance between a line and a plane

When calculating the distance between a line and a plane, first check if the line is parallel to the plane. If not, they intersect and the distance is zero. If they are parallel, find arbitrary point on the line and calculate its distance to the plane.

f) Projection of a line on the plane

To find the projection of the line AB on the plane α, we need a plane β containing the given line AB and perpendicular to the given plane α. The equation of the plane β and the plane α taken together are the equations of the projection.

g) Position of planes

Position of two planes

Consider two planes

$$\alpha \equiv a_1x + b_1y + c_1z = d_1$$
$$\beta \equiv a_2x + b_2y + c_2z = d_2$$

$\alpha \parallel \beta$ if their normal vectors are proportional i.e.

$(a_1, b_1, c_1) = k(a_2, b_2, c_2),\ \ k \in \mathbb{R}_0 \ \Rightarrow$

The two planes coincide if

$(a_1, b_1, c_1, d_1) = k(a_2, b_2, c_2, d_2),\ \ k \in \mathbb{R}_0$

That is, $\alpha = k\beta, k \in \mathbb{R}_0$. So $S = \alpha$ or $S = \beta$

The two planes are parallel and distinct if

$(a_1, b_1, c_1, d_1) \neq k(a_2, b_2, c_2, d_2),\ \ k \in \mathbb{R}_0$

and hence no intersection.

The two planes intersect , if their normal vectors are not proportional,

i.e. $(a_1,b_1,c_1)\neq k(a_2,b_2,c_2),\;\; k\in\mathbb{R}_0\;\Rightarrow\alpha\nparallel\beta$

The planes intersection is a line defined by the equations of the two planes taking together.

$$\begin{cases}a_1x+b_1y+c_1z=d_1\\a_2x+b_2y+c_2z=d_2\end{cases}$$

General equation of a line

The general equation of a straight line in space is

$$\begin{cases}a_1x+b_1y+c_1z=d_1\\a_2x+b_2y+c_2z=d_2\end{cases}$$

The direction vector of this line is

$$\left(\begin{vmatrix}b_1 & c_1\\b_2 & c_2\end{vmatrix},-\begin{vmatrix}a_1 & c_1\\a_2 & c_2\end{vmatrix},\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}\right)$$

Or to find the direction vector of the line, we can equate the right hand sides of the general equations to zero.

i.e,

$$\begin{cases}a_1x+b_1y+c_1z=0\\a_2x+b_2y+c_2z=0\end{cases}$$

Position of three planes

Consider three planes

$$\alpha\equiv a_1x+b_1y+c_1z=d_1$$
$$\beta\equiv a_2x+b_2y+c_2z=d_2$$
$$\gamma\equiv a_3x+b_3y+c_3z=d_3$$

There are three possible cases:

1. These planes are parallel if and only if the left hand sides of three equations are proportional.

   That is $(a_1,b_1,c_1)=k(a_2,b_2,c_2)$ and $(a_1,b_1,c_1)=m(a_3,b_3,c_3)$

In this case, the plane may be identical or distinct. We have two cases:

If $(a_1,b_1,c_1,d_1)=k(a_2,b_2,c_2,d_2)$, $(a_1,b_1,c_1,d_1)=m(a_3,b_3,c_3,d_3)$ and $(a_2,b_2,c_2,d_2)=n(a_3,b_3,c_3,d_3)$

The three equations are proportional and hence the three planes are coincident (identical), meaning that $\alpha \equiv \beta \equiv \gamma$

.

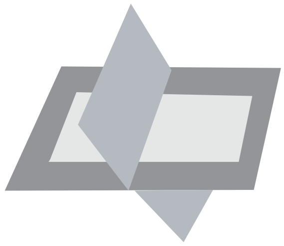
If $(a_1,b_1,c_1,d_1)\neq k(a_2,b_2,c_2,d_2)$ or
$(a_1,b_1,c_1,d_1)\neq m(a_3,b_3,c_3,d_3)$ or
$(a_2,b_2,c_2,d_2)\neq n(a_3,b_3,c_3,d_3)$

There are two equations that are not proportional but with proportional left hand sides and hence two planes are parallel and distinct and the third may be coincident to one of the other two or distinct to another. Then there is no intersection.

2. Two of them are parallel and the third is secant if and only if only two equations have the left hand sides that are proportional.

   In this case, there are two planes that are parallel and the third is secant.

If only two equations are proportional, two planes are coincident and the third is secant to them. Hence, the intersection is a straight line.



If the left hand sides of only two equations are proportional, two planes are parallel and distinct. Hence, no intersection.

3. No plane is parallel to another if and only if no left hand side of any equation is proportional to another.

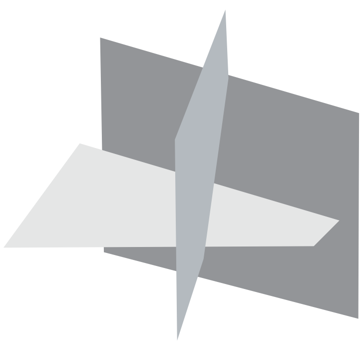
   a) There is one left hand side which is a linear combination of two others; in this case, there is a line of intersection of two planes which is parallel to the third.

      (i) If the corresponding equation is not a linear combination of two others, the line of intersection of two planes is strictly parallel to the third plane and hence there is no intersection between three planes.

(ii) If the corresponding equation is a linear combination of two others, the line is included in the third plane and hence this line is the intersection for three planes.

To find equation of the line of intersection, we proceed in the same way as for the case of two planes by taking any two equations from the three given equations of planes.

b) No left hand side is a linear combination of others, meaning that the three equations are linearly independent; in this case, the line of intersection of two planes pierces the third plane and hence there is a point of intersection between three planes.



To find this point, we solve simultaneously the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 8.11 Page 302

Materials

Exercise book, pens

Answers

1. Vector equation is

   $\overrightarrow{PV} = r\vec{u} + s\vec{v}$ or $\overrightarrow{OV} = \overrightarrow{OP} + r\vec{u} + s\vec{v}$. With $0(0,0,0)$ and r, s are parameters.

2. Parametric equations:

   From vector equation we have,

   $(x - x_0, y - y_0, z - z_0) = r(x_1, y_1, z_1) + s(x_2, y_2, z_2)$

Or

$$(x,y,z)=(x_0,y_0,y_0)+r(x_1,x_1,x_1)+s(x_2,y_2,z_2).$$

Thus the parametric equations are

$$\begin{cases} x=x_0+rx_1+sx_2 \\ y=y_0+ry_1+sy_2 \\ z=z_0+rz_1+sz_2 \end{cases}$$

3. Cartesian equation

$$(x-x_0)\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}-(y-y_0)\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}+(z-z_0)\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}=0$$

Let $\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}=a$, $-\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}=b$ and $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}=c$

We have:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Exercise 8.11 Page 306

1. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}+r\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}+s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

   Parametric equations: $\begin{cases} x=2+r+2s \\ y=4+3r+s \\ z=1-r+3s \end{cases}$

   Cartesian equation $10x-5y-5z=-5$

2. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+r\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}+s\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

   Parametric equations: $\begin{cases} x=1+4r-2s \\ y=1-2r+4s \\ z=1+r+3s \end{cases}$

   Cartesian equation $-10x-14y+12z=-12$

3. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 + r + 5s \\ y = 6 + s \\ z = r + 7s \end{cases}$

Cartesian equation $-x - 7y + z = -45$

4. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} + r\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 8 \\ 6 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 4 - 4r - 2s \\ y = 3 + r + 8s \\ z = 8 + r + 6s \end{cases}$

Cartesian equation $-2x + 22y - 30z = -182$

Activity 8.12 Page 306

Materials

Exercise book, pens

Answers

1. Vector equation is

$\overrightarrow{PX} = r\overrightarrow{PQ} + s\vec{v}$ or $\overrightarrow{OX} = \overrightarrow{OP} + r\overrightarrow{PQ} + s\vec{v}$, with $0(0,0,0)$, r and s are parameters

2. Parametric equations

$$\begin{cases} x = x_0 + r(x_1 - x_0) + sx_2 \\ y = y_0 + r(y_1 - y_0) + sy_2 \\ z = z_0 + r(z_1 - z_0) + sz_2 \end{cases}$$

3. Cartesian equation:

Eliminate the parameters in parametric equations or find the following determinant

$$\begin{vmatrix} x-x_0 & x_1-x_0 & x_2 \\ y-y_0 & y_1-y_0 & y_2 \\ z-z_0 & z_1-z_0 & z_2 \end{vmatrix}=0$$

$$(x-x_0)\begin{vmatrix} y_1-y_0 & y_2 \\ z_1-z_0 & z_2 \end{vmatrix}-(y-y_0)\begin{vmatrix} x_1-x_0 & x_2 \\ z_1-z_0 & z_2 \end{vmatrix}+(z-z_0)\begin{vmatrix} x_1-x_0 & x_2 \\ y_1-y_0 & y_2 \end{vmatrix}$$

Exercise 8.12 Page 309

1. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}+r\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}+s\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

   Parametric equations: $\begin{cases} x=2+s \\ y=4-3r+3s \\ z=1+2r-s \end{cases}$

   Cartesian equation $-3x+2y+3z=5$

2. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}+r\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}+s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

   Parametric equations: $\begin{cases} x=2-s \\ y=1+2s \\ z=-1+4r+s \end{cases}$

   Cartesian equation $-8x-4y=-20$

3. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+r\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}+s\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

   Parametric equations: $\begin{cases} x=1-3r+4s \\ y=1+3r-2s \\ z=1+2r+s \end{cases}$

   Cartesian equation $7x+11y-6z=12$

4. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + r\begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 + 2r + s \\ y = 6 - 5r \\ z = 7r + s \end{cases}$

Cartesian equation $-5x + 5y + 5z = 15$

Activity 8.13 Page 309

Materials

Exercise book, pens

Answers

1. Vector equation is

$\overrightarrow{PX} = r\overrightarrow{PQ} + s\overrightarrow{PN}$ or $\overrightarrow{0X} = \overrightarrow{0P} + r\overrightarrow{PQ} + s\overrightarrow{PN}$

2. Parametric equations

$$\begin{cases} x = x_0 + r(x_1 - x_0) + s(x_2 - x_0) \\ y = y_0 + r(y_1 - y_0) + s(y_2 - y_0) \\ z = z_0 + r(z_1 - z_0) + s(z_2 - z_0) \end{cases}$$

3. Cartesian equation:

$$(x - x_0)\begin{vmatrix} y_1 - y_0 & y_2 - y_0 \\ z_1 - z_0 & z_2 - z_0 \end{vmatrix} - (y - y_0)\begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ z_1 - z_0 & z_2 - z_0 \end{vmatrix}.$$

$$+(z - z_0)\begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix}$$

Exercise 8.13 Page 315

1. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + r\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 2 - r \\ y = 4 - r - 3s \\ z = r - r + 2s \end{cases}$

Cartesian equation: $-8x + 2y + 3z = -5$

2. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 1 + 3r - 3s \\ y = 1 - 3r + 3s \\ z = 1 + 2s \end{cases}$

Cartesian equation: $-x - y = -2$

3. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + r\begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 - 2r + 2s \\ y = 6 - 6r - 5s \\ z = r + 7s \end{cases}$

Cartesian equation: $-37x + 16y + 22z = -15$

4. Vector equation: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} + r\begin{pmatrix} -8 \\ -2 \\ -7 \end{pmatrix} + s\begin{pmatrix} -6 \\ 5 \\ -2 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 4 - 8r - 6s \\ y = 3 - 2r + 5s \\ z = 8 - 7r - 2s \end{cases}$

Cartesian equation: $3x + 2y - 4z = -14$

Activity 8.14 Page 319

Materials

Exercise book, pens, calculator

Answers

1. Equation of plane passing through points $(1,2,-1),(2,3,1),(3,-1,0)$

$$\begin{vmatrix} x & 1 & 2 & 3 \\ y & 2 & 3 & -1 \\ z & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

The fourth point $(1,2,1)$ lies on this plane if

$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} 2 & 3 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ -1 & 1 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow 7+6-5-18=-10$$

Thus, the four points do not lie on the same plane.

2. Equation of plane passing through points $(-2,1,1),(0,2,3),(1,0,-1)$

$$\begin{vmatrix} x & -2 & 0 & 1 \\ y & 1 & 2 & 0 \\ z & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

The fourth point $(2,1,-1)$ lies on this plane if

$$\begin{vmatrix} 2 & -2 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ -1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow 2\begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 0 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \end{vmatrix} = 0$$

$$\Leftrightarrow 0+10+5-5=10$$

Thus, the four points do not lie on the same plane.

3. Equation of plane passing through points $(1,0,-1),(0,2,3),(-2,1,1)$

$$\begin{vmatrix} x & 1 & 0 & -2 \\ y & 0 & 2 & 1 \\ z & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

The fourth point $(4,2,3)$ lies on this plane if

$$\begin{vmatrix} 4 & 1 & 0 & -2 \\ 2 & 0 & 2 & 1 \\ 3 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow 4\begin{vmatrix} 0 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 & -2 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$-\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0-20+15+5=0$$

Thus, the four points do lie on the same plane.

The four points

$(a_1,\ a_2,\ a_3)$; $(b_1,\ b_2,\ b_3)$;$(c_1,\ c_2,\ c_3)$ *and* $(d_1,\ d_2,\ d_3)$

lie on the same plane if the following condition is satisfied.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Exercise 8.14 Page 320

1. Not coplanar
2. $x = 4$
3. $a+b+c=2$
4. $a=-1,\ x-4y+3z-2=0$

Activity 8.15 Page 321

Materials

Exercise book, pens

Answers

1. a) $\overrightarrow{AX} = (x-a_1, y-a_2, z-a_3)$

   b) $a(x-a_1)+b(y-a_2)+c(z-a_3)=0$

   Expanding: $ax-aa_1+by-ba_2+cz-ca_3=0$ or

   $ax+by+cz=aa_1+ba_2+ca_3$.

   This is the equation of plane.
2. a) The line pierces the plane

   b) The line is parallel to the plane and lies in the plane

Exercise 8.15 Page 326

1. $3x-2y-z=-3$
2. $x+3y+4z=34$
3. Not parallel
4. $\begin{cases} x = 5-2t \\ y = 5t \\ z = -2+11t \end{cases}$

5. $(-1,0,0)$
6. The given line lies in the given plane
7. $\left(\frac{3}{2},-\frac{3}{2},\frac{1}{2}\right)$ 8. $(1,1,0)$
9. The given line lies in the given plane

Activity 8.16 Page 327

Materials

Exercise book, pens, calculator

Answers

1) 77.3^0 2) 35.6^0

Exercise 8.16 Page 330

1) $\frac{\pi}{4}$	2) 1.76 radians	3) 0.82 radians
4) 35^0	5) 79^0	6) 45^0
7) 67.09^0	8) 30^0	

Activity 8.17 Page 331

Materials

Exercise book, pens

Answers

Then the normalized normal vector $\vec{e}$ is

$$\vec{e}=\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

$$\overrightarrow{AB}=(b_1-a_1,b_2-a_2,b_3-a_3)$$

$$\left|\vec{e}\cdot\overrightarrow{AB}\right|=\left|\frac{a(b_1-a_1)}{\sqrt{a^2+b^2+c^2}}+\frac{b(b_2-a_2)}{\sqrt{a^2+b^2+c^2}}+\frac{c(b_3-a_3)}{\sqrt{a^2+b^2+c^2}}\right|$$

$$\Leftrightarrow\left|\vec{e}\cdot\overrightarrow{AB}\right|=\frac{\left|a(b_1-a_1)+b(b_2-a_2)+c(b_3-a_3)\right|}{\sqrt{a^2+b^2+c^2}}$$

$$\Leftrightarrow \left|\vec{e}\cdot\overrightarrow{AB}\right| = \frac{\left|ab_1 - aa_1 + bb_2 - ba_2 + cb_3 - ca_3\right|}{\sqrt{a^2+b^2+c^2}}$$

$$\Leftrightarrow \left|\vec{e}\cdot\overrightarrow{AB}\right| = \frac{\left|ab_1 + bb_2 + cb_3 - \left(aa_1 + ba_2 + ca_3\right)\right|}{\sqrt{a^2+b^2+c^2}}$$

Since $A \in \alpha$, $aa_1 + ba_2 + ca_3 = d$, we have

$$\left|\vec{e}\cdot\overrightarrow{AB}\right| = \frac{\left|ab_1 + bb_2 + cb_3 - d\right|}{\sqrt{a^2+b^2+c^2}}$$

Exercise 8.17 Page 335

1. 3 unit of length
2. $\frac{19}{5}$ unit of length
3. $\frac{5}{3}$ unit of length
4. $\frac{9\sqrt{41}}{41}$ unit of length
5. $\frac{5\sqrt{6}}{18}$ unit of length

Activity 8.18 Page 335

Materials

Exercise book, pens

Answers

Point on the line: $(2,3,1)$

Direction vector of the line: $(1,-2,1)$

Normal vector of the given plane $(2,3,-2)$

Normal vector of the needed plane, say (a,b,c), is perpendicular to the direction vector of the line and also perpendicular to the normal vector of the given plane.

The needed plane has the form:

$$a(x-2)+b(y-3)+c(z-1)=0$$

Where

$$\begin{cases} a-2b+c=0 \\ 2a+3b-2c=0 \end{cases}$$

Solving, we get

$$\begin{cases} a=\dfrac{1}{7} \\ b=\dfrac{4}{7} \\ c=1 \end{cases}$$

And the needed plane is $x+4y+7z-21=0$

Exercise 8.18 Page 337

1. $\dfrac{x}{2}=\dfrac{y-3}{-6}=\dfrac{z-6}{-5}$

2. $\begin{cases} 2x-3y+z-30=0 \\ 5x+4y+2z-15=0 \end{cases}$

Activity 8.19 Page 337

Materials

Exercise book, pens

Answers

$$\frac{x-1}{1}=\frac{y-2}{-4}=\frac{z-1}{-2}$$

Intersection: $(2,-2,1)$

Exercise 8.19 Page 340

1. $(-1,-2,1)$

2. $(2,-2,-1)$

Activity 8.20 Page 341

Materials

Exercise book, pens

Answers

1. $\vec{u}=(3,-2,1)$ 2. Point: $(-3,4,1)$ 3. $\begin{cases} x=-3+3t \\ y=4-2t \\ z=1+t \end{cases}$

Exercise 8.20 Page 348

1. a) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$ b) $\frac{3x+1}{9}=\frac{3y+2}{-6}=\frac{z}{1}$

   c) $\frac{x+2}{1}=\frac{y+3}{2}=\frac{z}{1}$

2. a) $(2,-5,3)$ b) $(a,1,c)$

3. $\overrightarrow{n_1}=\vec{i}-\vec{j}$, $\overrightarrow{n_2}=\vec{i}+\vec{j}+\vec{k}$

   The planes intersect in a line

   $\vec{u}=-\vec{i}-\vec{j}+2\vec{k}$

4. $3x-3y+4z+2=0$ 5. $x-5y-3z=-7$

6. $x+6y-5z=17$ 7. $4x-2y+7z=0$

Activity 8.21 Page 349

Materials

Exercise book, pens

Answers

The equation of plane α and plane γ are proportional. Then plane α and plane γ coincide.

The equation of plane β is not proportional to any other equation. Also the left hand side of the equation of plane β is not proportional to any other left hand sides of other equations, then plane β is secant to other planes.

Exercise 8.21 Page 357

1. The given planes coincide 2. No intersection
3. Point: $(1,2,3)$ 4. Point: $(-4,-3,0)$
5. No intersection

8.4. Sphere in three dimensions

Recommended teaching periods: 9 periods

This section looks at:

a) Equations of sphere in space

The equation of a sphere of centre (k,l,m) and radius r is given by

$S \equiv (x-k)^2 + (y-l)^2 + (z-m)^2 = r^2$

The general equation of a sphere is:

$x^2 + y^2 + z^2 + ax + by + cz + d = 0$

In this equation:

The centre is $\Omega = \left(-\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right)$ and the radius is given by

$r = \frac{1}{2}\sqrt{a^2 + b^2 + c^2 - 4d}$, provided that $a^2 + b^2 + c^2 - 4d > 0$

b) Position of a point and sphere

Consider a sphere S with radius r and centre $\Omega(a,b,c)$ and any point $P(a_1, a_2, a_3)$.

- If $d(\Omega, P) < r$, the point lies inside the sphere S.
- If $d(\Omega, P) = r$, the point lies on the sphere S.
- If $d(\Omega, P) > r$, the point lies outside the sphere S.

In all cases, $d(\Omega, P)$ is the distance between point P and centre Ω of sphere S.

c) Position of a sphere and a line

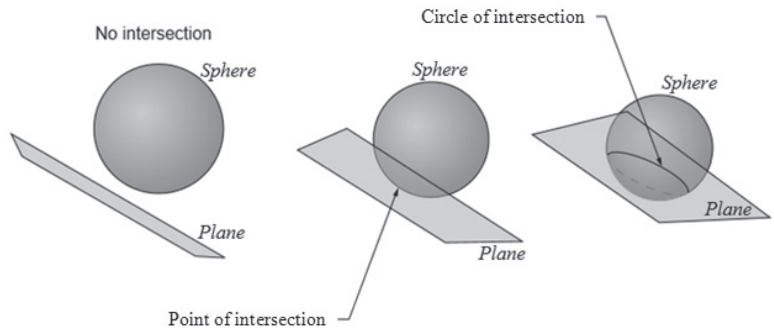
Consider a sphere S with radius r and centre $\Omega(a,b,c)$ and line L.

- If $d(\Omega, L) < r$, there are two points of intersection.
- If $d(\Omega, L) = r$, there is a single point of intersection.
- If $d(\Omega, L) > r$, there is no intersection.

In all cases, $d(\Omega, L)$ is the shortest distance between line L and centre Ω of sphere S.

d) Position of sphere and a plane

Consider a sphere $S \equiv x^2 + y^2 + z^2 + ax + by + cz + d = 0$ with centre $\Omega = (k, l, m)$ and radius r and plane $\alpha \equiv hx + ny + pz = q$, their position appears in three cases.



1. If $d(\Omega, \alpha) < r$, the plane cuts the sphere and the intersection is a circle whose centre is on the plane (on the perpendicular line of the plane passing through the centre of the sphere). When the plane cuts the sphere, we call it plane section of a sphere.

   Then, $d(P, Q) = \sqrt{\left[d(\Omega, Q)\right]^2 - \left[d(\Omega, P)\right]^2}$.

2. If $d(\Omega, \alpha) > r$, there is no intersection.
3. If $d(\Omega \quad) = r$, the plane is tangent to the sphere and the intersection is the point which lies on the perpendicular line of the plane passing through the centre of the sphere and it is the intersection between this perpendicular line and the plane.

In all cases, $d(\Omega, \alpha)$ is the distance between the centre $\Omega = (k, l, m)$ of the sphere $S \equiv x^2 + y^2 + z^2 + ax + by + cz + d = 0$ with radius

r and plane $\alpha \equiv hx+ny+pz=q$. It is given by

$$d(\Omega,\alpha)=\frac{|hk+nl+pm-q|}{\sqrt{h^2+n^2+p^2}}.$$

The tangent plane at $T(x_1,y_1,z_1)$ on the sphere $S \equiv x^2+y^2+z^2+ax+by+cz+d=0$ is

$$xx_1+yy_1+zz_1+a(x+x_1)+(y+y_1)+c(z+z_1)+d=0.$$

Hint. In writing equation of tangent plane to a sphere at a given point $T(x_1,y_1,z_1)$, in the sphere equation, change x^2 to xx_1, y^2 to yy_1, z^2 to zz_1, x to $\frac{1}{2}(x+x_1)$, y to $\frac{1}{2}(y+y_1)$, and z to $\frac{1}{2}(z+z_1)$ and then expand.

e) Position of two spheres

Consider two spheres with centers Ω_1 and Ω_2; radii r_1 and r_2. The position of these two spheres depends on the distance between theirs centers, $d(\Omega_1,\Omega_2)$

- If $d>r_1+r_2$. Two spheres are exterior and hence no intersection.
- If $d<r_1+r_2$. Two spheres are interior and hence no intersection.
- If $d=r_1+r_2$. Two spheres are tangent exterior and hence there is a point of intersection.
- If $d=|r_1-r_2|$. Two spheres are tangent interior and hence there is a point of intersection.
- If $|r_1-r_2|<d<r_1+r_2$. One sphere cuts another. The intersection is a circle.

Teaching guidelines

Let learners know what is a circle in 2-dimensions. When we rotate a half circle about $x-axis$ we obtain a sphere.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 8.22 Page 358

Materials

Exercise book, pens

Answers

1. $(x-k)^2+(y-l)^2+(z-m)^2=r^2$

$\Leftrightarrow x^2-2kx+k^2+y^2-2ly+l^2+z^2-2mz+m^2=r^2$

$\Leftrightarrow x^2+y^2+z^2-2kx-2ly-2mz+k^2+l^2+m^2-r^2=0$

2. Letting $-2k=a, -2l=b, -2m=c,\ k^2+l^2+m^2-r^2=d$

Gives $k=-\frac{a}{2}, l=-\frac{b}{2}, m=-\frac{c}{2}$ and $-r^2=-k^2-l^2-m^2+d$

Or

$r^2=k^2+l^2+m^2-d \qquad r^2=\left(-\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2+\left(-\frac{c}{2}\right)^2-d$

$r^2=\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}-d \qquad r^2=\frac{a^2+b^2+c^2-4d}{4}$

$r=\frac{1}{2}\sqrt{a^2+b^2+c^2-4d}$

Exercise 8.22 Page 363

1. a) $x^2+y^2+z^2-2x-4y-6z-2=0$

   b) $x^2+y^2+z^2-6x+2y-2z+8=0$

   c) $x^2+y^2+z^2-8x+2z-32=0$

2. a) $(11,3,0)$; 8 b) $(-4,8,7)$; 6 c) $(0,9,3)$; 14

3. a) The interior of the sphere $x^2+y^2+z^2=4$.

   b) The solid ball bounded by the sphere $(x+2)^2+(y-3)^2+(z+4)^2=4$. Alternatively, the sphere $(x+2)^2+(y-3)^2+(z+4)^2=4$ together with its interior.

   c) The exterior of the sphere $(x-1)^2+(y+3)^2+z^2=8$.

Activity 8.23 Page 363

Materials

Exercise book, pens, calculator

Answers

1. $\sqrt{38}$, outside the sphere 2. $\sqrt{6}$, on the sphere
3. $\sqrt{5}$, inside the sphere

Exercise 8.23 Page 365

1. Outside the sphere 2. Inside the sphere
3. On the sphere 4. Outside the sphere

Activity 8.24 Page 365

Materials

Exercise book, pens, calculator

Answers

1. $\frac{3\sqrt{2}}{2}$, the line pierces the sphere
2. $2\sqrt{42}$, the line does not touch the sphere
3. $\frac{\sqrt{3}}{2}$, the line is tangent to the sphere

Exercise 8.24 Page 368

1. $x^2+y^2+z^2-6\sqrt{2}(x+y+z)-3\sqrt{2}=0$
2. $(1,-1,3)$, $(5,2,-2)$

Activity 8.25 Page 369

Materials

Exercise book, pens, calculator

Answers

1. $\frac{\sqrt{14}}{2}$, the plane cuts the sphere
2. $\frac{8\sqrt{14}}{7}$, the plane does not touch the sphere
3. $\sqrt{14}$, the plane is tangent to the sphere

Exercise 8.25 Page 377

1. $2x+2y-z+10=0,\ 2x+2y-z-8=0$
2. $2x+y-2z=9,\ x+2y+2z=9$
3. $x^2+y^2+z^2-6x-4y-2z+5=0$,

   $x^2+y^2+z^2-6x-\frac{11}{4}y-2z+5=0$
4. $x^2+y^2+z^2-2x+2y-4z+2=0$,

   $x^2+y^2+z^2-6x-4y+10z+22=0$

Activity 8.26 Page 378

Materials

Exercise book, pens, calculator

Answers

Let centers of two spheres be Ω_1 and Ω_2 respectively and their radii be r_1 and r_2 respectively

1. $d(\Omega_1,\Omega_2)=\sqrt{11}$ and $r_1+r_2=5$. One sphere is inside of another
2. $d(\Omega_1,\Omega_2)=9$ and $r_1+r_2=9$. One sphere touches another
3. $d(\Omega_1,\Omega_2)=\sqrt{46}$ and $r_1+r_2=\sqrt{3}+\sqrt{4}$. One sphere is outside of another

Exercise 8.26 Page 389

1. $x^2+y^2+z^2+3y+5z-7=0$
2. $x^2+y^2+z^2-2z-8=0$
3. $13\left(x^2+y^2+z^2\right)-35x-21y+43z+176=0$

End of Unit Assessment Page 398

1. a) $\frac{1}{3}\vec{A}+\frac{2}{3}\vec{B}$ b) $\frac{3}{7}\vec{A}+\frac{4}{7}\vec{B}$ c) $\frac{3}{5}\vec{A}+\frac{2}{5}\vec{B}$

   d) $\frac{3}{2}\vec{A}-\frac{1}{2}\vec{B}$ e) $-\frac{2}{3}\vec{A}+\frac{5}{3}\vec{B}$ f) $4\vec{A}-3\vec{B}$
2. $\overrightarrow{OM}=\frac{1}{2}\left(\vec{b}+\vec{c}\right)$ $\overrightarrow{ON}=\frac{1}{3}\left(\vec{a}+\vec{b}+\vec{c}\right)$
3. $x\vec{i}+y\vec{j}+z\vec{k}=\vec{i}+\vec{j}+\vec{k}+\lambda\left(2\vec{i}+3\vec{j}-\vec{k}\right)$
4. $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$

5. $4\vec{i}-\vec{j}+12\vec{k}=2\vec{i}+3\vec{j}+4\vec{k}+r\left(\vec{i}-2\vec{j}+4\vec{k}\right)$

$2\vec{i}-4\vec{j}+8\vec{k}=r\left(\vec{i}-2\vec{j}+4\vec{k}\right)$

$$\begin{cases} 2=r & \Rightarrow r=2 \\ -4=-2r & \Rightarrow r=2 \\ 8=4r & \Rightarrow r=2 \end{cases}$$

Thus the given point lie on the given line.

6. $a=6,\ b=8$

7. a) $(3,3,-3)$ b) $(x,y,z)=(2,-1,1)+\lambda(3,3,-3)$

8. a) $\frac{x-2}{2}=\frac{y-3}{3}=\frac{z+1}{1}$ b) $\frac{x-3}{3}=\frac{y+1}{2}=\frac{z-2}{-4}$

c) $\frac{x-2}{2}=\frac{y-1}{-1}=\frac{z-1}{-1}$

9. $(x,y,z)=(2,5,4)+\lambda(3,-2,-1)$

10. a) $(x,y,z)=(2,2,-1)+\lambda(3,2,4)$

b) $(x,y,z)=(3,-2,3)+\lambda(1,4,-1)$

11. $(2,5,3)$ or its multiples

12. Vector equation: $(x,y,z)=(1,0,-3)+\lambda(2,-1,3)$

Parametric equations: $\begin{cases} x=1+2\lambda \\ y=-\lambda \\ z=-3+2\lambda \end{cases}$

Symmetric equations: $\frac{x-1}{2}=-y=\frac{z+3}{3}$

13. a) $(x,y,z)=(2,3,4)+\lambda(2,-3,2)+\mu(0,1,2)$

b) $(x,y,z)=(0,0,-2)+\lambda(3,3,-1)+\mu(1,-1,1)$

c) $(x,y,z)=(-2,-1,-3)+\lambda(1,0,1)+\mu(2,1,1)$

d) $(x,y,z)=(5,1,-4)+\lambda(1,-1,1)+\mu(3,-1,-1)$

14. a) $2x+3y-4z=29$ b) $2x+y-4z=12$

c) $x+y+z=0$ d) $3x+4y-5z=12$

e) $2x-y=4$ f) $5y+2z=-11$ g) $x=3$

15. $x+5y-4z=20$

16. $x-3y-6z+8=0$

17. $5x-6y+7z=20$

18. $x+2y-3z=0$

19. $\begin{pmatrix}7\\-5\\-4\end{pmatrix}=\begin{pmatrix}4\\3\\2\end{pmatrix}+r\begin{pmatrix}1\\-1\\-1\end{pmatrix}+s\begin{pmatrix}2\\3\\1\end{pmatrix}\Rightarrow\begin{cases}r=5\\s=-1\end{cases}$. Thus, the given point lie on the given plane.

20. $2x-3y-6z=6$, $6x+3y-2z=18$

21. $(x,y,z)=(1,0,-2)+\lambda(1,1,0)+\mu(0,0,1)$

22. a) $\sqrt{2}$ units of length b) 3 units of length

    c) $\sqrt{10}$ units of length d) $\frac{1}{2}\sqrt{138}$ units of length

23. a) 7 units of length; $(1,2,3)$

    b) $2\sqrt{6}$ units of length; $(2,-1,-1)$

    c) 6 units of length; $(4,0,0)$

    d) $\frac{1}{3}\sqrt{42}$ units of length; $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$

24. 14 units of length

25. 38.31 degrees

26. 25.7 degrees

27. 80 degrees

28. 45.6 degrees

29. 40.2 degrees

30. 90 degrees

31. a) $(3,-4,5)$; 7 b) $(1,2,3)$; 3

    c) $\left(\frac{1}{2},-1,-\frac{1}{2}\right)$, 1

32. a) $x^2+y^2+z^2-2x-4y-6z+10=0$

    b) $x^2+y^2+z^2-4x-4z+4=0$

    c) $x^2+y^2+z^2-4x-6y-23=0$

33. $d=-3$

34. $7\left(x^2+y^2+z^2\right)-15x-25y-11z=0$

35. $x^2+y^2+z^2-4x+6y-2z+5=0$

36. $3\left(x^2+y^2+z^2\right)-2x-2y-2z-1=0$

37. $x^2+y^2+z^2-6x-4y+10z+12=0$

38. $x^2+y^2+z^2-x-6y-2z+5=0$

39. $5\left(x^2+y^2+z^2\right)-2x-2y-2z-9=0$, $\left(\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$, $\frac{4\sqrt{3}}{2}$

40. $x^2+y^2+z^2+7y-8z+24=0$

41. $x^2+y^2+z^2-4x+6y-8z+4=0$

42. $x^2+y^2+z^2+2x+2y+2z+2=0$

43. $x^2+y^2+z^2\pm 6z-4=0$

44. $4x+15y+26z-75=0$

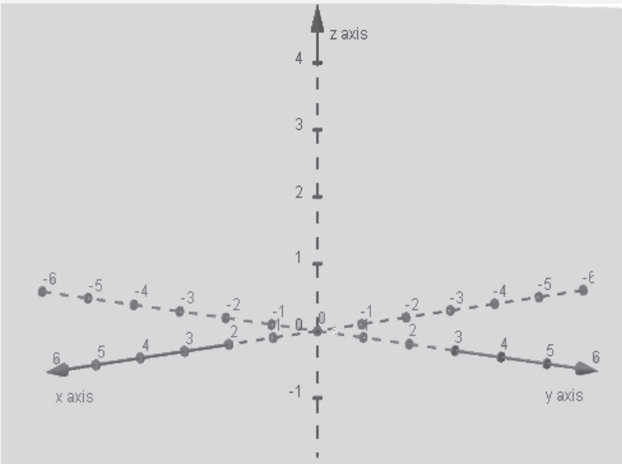
45. -4, 14

46. $(4,-2,2),(0,0,-2)$

47. $9\left(x^2+y^2+z^2\right)=5$

48. $\sqrt{61}$

Sketch



49. $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$

Unit 9 Bivariate Statistics

Learner's Book page 405 – 434

Aim

Extend understanding, analysis and interpretation of bivariate data to correlation coefficients and regression lines

Objectives

After completing this unit, the learners should be able to:

- find the covariance of two quantitative variables.
- determine the linear regression line of a given series.
- calculate a linear correlation coefficient of a given double series and interpret it.

Materials

Exercise books, pens, calculator

Contents

9.1. Covariance

Recommended teaching periods: 2 periods

This section looks at formula used to find the covariance of two variable *x* and *y*.

$$\text{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y}$$

Teaching guidelines

Let learners know how to find variance and standard deviation of a series. In bivariate statistics, we use two series.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.

- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 9.1 Page 406

Materials

Exercise book, pens, calculator

Answers

x	y	$x-\overline{x}$	$y-\overline{y}$	$(x-\overline{x})(y-\overline{y})$
3	6	-1.3	-2.6	-3.9
5	9	0.7	0.4	1.1
7	12	2.7	3.4	6.1
3	10	-1.3	1.4	0.1
2	7	-2.3	-1.6	-3.9
6	8	1.7	-0.6	1.1
$\sum_{i=1}^{6} x_i = 26$	$\sum_{i=1}^{6} y_i = 52$			$\sum_{i=1}^{6}(x_i-\overline{x})(y_i-\overline{y}) = 0.6$
$\overline{x} = 4.3$	$\overline{y} = 8.6$			

1. If you divide by total frequency you get **variance**
2. If you divide by total frequency you get **covariance**

Exercise 9.1 Page 410

1. $\operatorname{cov}(x,y)=\frac{71}{12}$	2. $\operatorname{cov}(x,y)=98.75$

9.2. Regression lines

Recommended teaching periods: 5 periods

This section looks at the adjustment of algebraic expression of two regression lines.

The regression line *y* on *x* is written as

$$L_{y/x} \equiv y-\overline{y}=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}\left(x-\overline{x}\right)$$

The regression line x on y is written as

$$L_{x/y} \equiv x-\overline{x}=\frac{\operatorname{cov}(x,y)}{\sigma_y^2}\left(y-\overline{y}\right)$$

Teaching guidelines

Let learners know how to find mean, standard deviation and covariance.

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 9.2 Page 411

Materials

Exercise book, pens, calculator

Answers

1. $D_b^{/}=2\sum_{i=1}^{k}(y_i-ax_i-b)(-1)$ or $D_b^{/}=-2\sum_{i=1}^{k}(y_i-ax_i-b)$

2. $\sum_{i=1}^{k}(y_i-ax_i-b)=0$ or $\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}ax_i-\sum_{i=1}^{k}b=0$

   or $\sum_{i=1}^{k}b=\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}ax_i$

   Dividing both sides by n gives

   $\frac{1}{n}\sum_{i=1}^{k}b=\frac{1}{n}\sum_{i=1}^{k}y_i-\frac{1}{n}\sum_{i=1}^{k}ax_i$ or $\frac{b}{n}\sum_{i=1}^{k}1=\frac{1}{n}\sum_{i=1}^{k}y_i-\frac{a}{n}\sum_{i=1}^{k}x_i$

   or $b=\overline{y}-a\overline{x}$

3. $\sum_{i=1}^{k}(y_i-ax_i-b)^2=\sum_{i=1}^{k}\left(y_i-ax_i-\overline{y}+a\overline{x}\right)^2$

   Or

   $\sum_{i=1}^{k}(y_i-ax_i-b)^2=\sum_{i=1}^{k}\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]^2$

   Differentiation with respect to a and equating to zero:

   $\sum_{i=1}^{k}2\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]\left[-\left(x_i-\overline{x}\right)\right]=0$

   $-2\sum_{i=1}^{k}\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]\left(x_i-\overline{x}\right)=0$

   $\Leftrightarrow\sum_{i=1}^{k}\left[\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)\right]\left(x_i-\overline{x}\right)=0$

   $\Leftrightarrow\sum_{i=1}^{k}\left[\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)-a\left(x_i-\overline{x}\right)^2\right]=0$

$$\Leftrightarrow \sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)-\sum_{i=1}^{k}a\left(x_i-\overline{x}\right)^2=0$$

$$\Leftrightarrow \sum_{i=1}^{k}a\left(x_i-\overline{x}\right)^2=\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

$$\Leftrightarrow a\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2=\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

Dividing both sides by n gives

$$\Leftrightarrow \frac{a}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

$$\Rightarrow a=\frac{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)}{\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2}$$

4. The variance for variable x is $\sigma_x^2=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)^2$

   and the variance for variable y is $\sigma_y^2=\frac{1}{n}\sum_{i=1}^{k}\left(y_i-\overline{y}\right)^2$

   and the covariance of these two variables is

   $$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k}\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$$

   Then $a=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}$

5. Now, we have that the regression line y on x is $y=ax+b$, where

   $$\begin{cases} a=\dfrac{\operatorname{cov}(x,y)}{\sigma_x^2} \\ b=\overline{y}-a\overline{x} \end{cases}$$

   Or

   $$y=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}x+\left(\overline{y}-\frac{\operatorname{cov}(x,y)}{\sigma_x^2}\overline{x}\right)$$

Exercise 9.2 Page 416

1. a) $y = 0.19x - 8.098$ b) $y = 4.06$
2. $x = -5.6y + 163.3, \quad y = -0.06x + 21.8$

9.3. Coefficient of correlation

Recommended teaching periods: 3 periods

This section looks at the correlation coefficient and its properties. It also looks at the Spearman's coefficient of rank correlation.

- **The Pearson's correlation coefficient**,

  Pearson's correlation coefficient denoted by r, is a measure of the strength of linear relationship between two variables.

  The correlation coefficient between two variables x and y is given by $r = \dfrac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$.

- **The Spearman's coefficient of rank correlation**

  The Spearman's coefficient of rank correlation is given by

  $$\rho = 1 - \frac{6\sum_{i=1}^{k} d_i^2}{n(n^2 - 1)}$$

  Where, d refers to the difference of ranks between paired items in two series.

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
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Answers to activities and exercises

Activity 9.3 Page 416

Materials

Exercise book, pens, calculator

Answers

1. $\sigma_x = 1.8, \sigma_y = 1.97$
2. $\operatorname{cov}(x, y) = \frac{41}{18}$
3. $\frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y} = 0.64$

Exercise 9.3 Page 426

1. $r = 0.94$. As the correlation coefficient is very close to 1, the correlation is very strong.
2. $r = -0.26$. As the correlation coefficient is very close to zero, the correlation is very weak.
3. $\sigma = 0.14$. There is a weak positive correlation between the English and Mathematics rankings.
4. $\sigma = 1$. There is a perfect agreement between the rankings.

9.4. Applications

Activity 9.4 Page 427

Materials

Exercise book, pens, calculator

Answers

By reading textbooks or accessing internet, learners will discuss how bivariate statistics is used in daily life. Bivariate statistics can help in prediction of a value for one variable if we know the value of the other by using regression lines.

End of Unit Assessment Page 429

1. Data set 1

   a) $y = 4.50 + 0.64x$ b) $x = 4.42 + 0.75y$

   Data set 2

   a) $y = 90.31 - 1.78x$ b) $x = 37.80 - 0.39y$

2. $y = -2.59 + 0.65x; 36.5$
3. $r = 0.918, y - 65.45 = 0.981(x - 65.18), x - 65.18 = 0.859(y - 65.45)$
4. $y = 0.611x + 10.5, x = 1.478y - 1.143, y = 28.83$
5. $y = 0.94x + 92.26, Blood\ \ pressure = 134.56$
6. $y = 3.8 + 1.6x, x = -2.06 + 0.59y$
7. $y = -8 + 1.2x$
8. $c = 15, d = -5$
9. 0.60, $w = 0.89h - 76$
10. $\bar{x} = -\frac{3}{29}, \bar{y} = \frac{15}{29}, r = \frac{3}{4}$
11. $r = 0.4$
12. 0.82
13. 0.77

14. -0.415

15. a) 0.954 b) $\bar{x} = 2$, $\bar{y} = 3$

16. $\bar{x} = 13, \bar{y} = 17, y = 0.8x + 6.6, x = 0.45y - 5.35, r = 0.6, \sigma_y = 4$

17. $\bar{x} = 13, \bar{y} = 17, \sigma_y = 4$

18. 0.26

19. a) 0.43

b) Some agreement between average attendances ranking a position in league, high position in league correlating with high attendance.

20. a) (i) -0.976 (ii) -0.292 (or 0.292)

b) The transport manager's order is more profitable for the seller, saleswomen is unlikely to try to dissuade.

c) (i) No, maximum value is 1

(ii) Yes, higher performing cars generally do less mileage to the gallon.

(iii) No, the higher the engine capacity, the dearer the car.

d) When only two rankings are known; when relationship is non-linear.

21. a) There is a strong positive correlation

b) 54.5

Unit 10 Conditional Probability and Bayes Theorem

Learner's Book page 435 – 454

Aim

Solve problems using Bayes theorem and use data to make decisions about likelihood and risk.

Objectives

After completing this unit, the learners should be able to:

- use tree diagram to find probability of events.
- find probability of independent events.
- find probability of one event given that the other event has occurred.
- use and apply Bayes theorem.

Materials

Exercise books, pens, ruler, calculator

Contents

10.1. Tree diagram

Recommended teaching periods: 5 periods

This section shows the method used to find probability of events by constructing tree diagram.

A **tree diagram** is a means which can be used to show the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession.

The **outcome** is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring.

For each **trial,** the number of branches is equal to the number of possible outcomes of that trial. In the diagram, there are two possible outcomes, *A* and *B*, of each trial.

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 10.1 Page 436

Materials

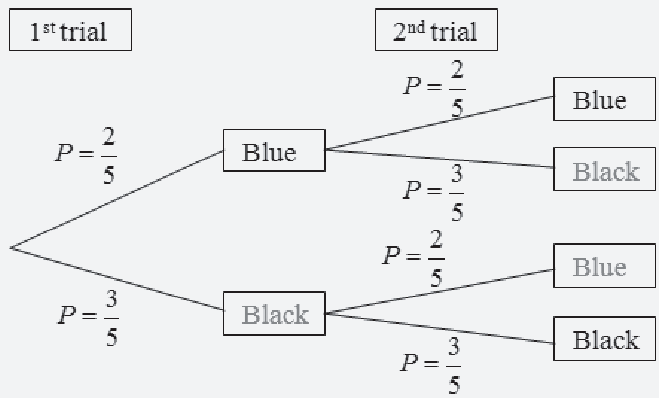
Exercise book, pens, calculator

Methodology

Facilitate learners in Group work, then questioning.

Answers

1. Probability of choosing a blue pen is $\frac{4}{10}=\frac{2}{5}$ and probability of choosing a black pen is $\frac{6}{10}=\frac{3}{5}$.
2. Probabilities on the second trial are equal to the probabilities on the first trial since after the 1st trial the pen is replaced in the box.
3. Complete figure



Exercise 10.1 Page 439

1. $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

2. a) $P(3\,\text{boys})=\frac{10}{16}\times\frac{9}{15}\times\frac{8}{14}=0.214$

   b) $P(2\text{ boys and 1 girl})=\frac{10}{16}\times\frac{9}{15}\times\frac{6}{14}+\frac{10}{16}\times\frac{6}{15}\times\frac{9}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{9}{14}=0.482$

   c) $P(2\text{ girls and 1 boy})=\frac{10}{16}\times\frac{6}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{10}{15}\times\frac{5}{14}+\frac{6}{16}\times\frac{5}{15}\times\frac{10}{14}=0.268$

   d) $P(3\,\text{girls})=\frac{6}{16}\times\frac{5}{15}\times\frac{4}{14}=0.0357$

3. a) $\frac{1}{21}$ b) $\frac{10}{21}$ c) $\frac{11}{21}$

4. a) $\frac{1}{816}$ b) $\frac{7}{102}$ c) $\frac{7}{34}$

10.2. Independent events

Recommended teaching periods: 5 periods

This section shows the formula used to find probability of independent events.

If probability of event B is not affected by the occurrence of event A, events A and B are said to be independent and $P(A \cap B) = P(A) \times P(B)$

This rule is the simplest form of the multiplication law of probability.

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework

Answers to activities and exercises

Activity 10.2 Page 440

Materials

Exercise book, pens

Answers

The occurrence of event B does not affected by occurrence of event A because after the first trial, the pen is replaced in the box. It means that the sample space does not change.

Exercise 10.2 Page 442

1. $P(\text{red and red}) = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$
2. $P(\text{head and 3}) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$
3. a) $\frac{1}{35}$ b) $\frac{2}{7}$

10.3. Conditional probability

Recommended teaching periods: 5 periods

This section shows the formula used to find conditional probability.

The probability of an event B given that event A has occurred is called the conditional probability of B given A and is written $P(B \mid A)$.

In this case, $P(B \mid A)$ is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A \cap B$, then

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}.$$

From this result, we have a general statement of the multiplication law:

$$P(A \cap B) = P(A) \times P(B \mid A)$$

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.
- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 10.3 Page 443

Materials

Exercise book, pens

Answers

The occurrence of event *B* is affected by occurrence of event *A* because after the first trial, the pen is not replaced in the box. It means that the sample space will be changed for the second trial.

Exercise 10.3 Page 447

1. $P(6 \mid even) = \dfrac{\frac{1}{6}}{\frac{3}{6}} = \dfrac{1}{3}$

2. $P(White \mid Black) = \dfrac{P(Black \text{ and } White)}{P(Black)} = \dfrac{0.34}{0.47} = 0.72$

3. $\dfrac{}{13}$

10.4. Bayes theorem and applications

Recommended teaching periods: 5 periods

This section shows how to find probability of events using Bayes theorem and its applications.

Let $B_1, B_2, B_3, ..., B_n$ be incompatible and exhaustive events and let A be an arbitrary event.

We have:

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

This formula is called Bayes' formula.

Remark

We also have (Bayes' rule)

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

Teaching guidelines

- Organise class into groups. Request each group to have a group leader who will present their findings to the class.
- Ask each group to do activity related to the lesson they are going to learn. The learners may need your assistance to do any activity. Help them to understand the activity. After group discussion, invite some or all groups for presentation of their work.

- After activity presentation, capture the main points from the presentation of the learners and summarise them. Guide the learners through given examples in Learner's book or through your own examples.
- Ask learners what they learned in day lesson to ensure that they understood what they have learned.
- Request learners to do exercises in their respective groups. Request learners to correct exercises on chalkboard and give them individual evaluation. Remember to give them homework.

Answers to activities and exercises

Activity 10.4 Page 448

Materials

Exercise book, pens

Answers

1. $$P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)$$

2. $$P(B_1|A)=\frac{P(B_1\cap A)}{P(A)}=\frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_2|A)=\frac{P(B_2\cap A)}{P(A)}=\frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_3|A)=\frac{P(B_3\cap A)}{P(A)}=\frac{P(A|B_3)P(B_3)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

Generally,

$$P(B_i|A)=\frac{P(B_i\cap A)}{P(A)}=\frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{3}P(A|B_i)P(B_i)}$$

Exercise 10.4 Page 450

1. $P(engineer \mid managerial) = \dfrac{0.2\times 0.75}{0.2\times 0.75 + 0.2\times 0.5 + 0.6\times 0.2} = 0.405$

2. $P(No\ accident \mid Triggered\ alarm) = \dfrac{0.9\times 0.02}{0.1\times 0.97 + 0.9\times 0.02} = 0.157$

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1. 0.15
2. 0.13
3. 0.56
4. $\frac{1}{169}$
5. $\frac{15}{128}$
6. $\frac{729}{1000}$
7. 0.37
8. $\frac{10}{21}$
9. a) 0.34 b) 0.714 c) 0.0833
10. a) 0.43 b) 0.1166 c) 0.8966
11. a) 0.0001 b) 0.0081
12. a) 0.384 b) 0.512
13. 0.1083
14. a) 0.5514 b) 0.2941
15. $\frac{3}{13}$

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